STUDY GUIDE FOR THE FINAL EXAM OF MATH 2568
SPRING 2020 - PROF. CUETO

The final exam is **cummulative** and will cover basic topics on Linear algebra. Topic 1 was covered for Midterm 1, Midterm 2 material included Topics 2–4. The rest of the topics were taught after Spring Break (in online fashion).

The following is a cheat sheet containing the main results in each topic, with references and where to find them.

Topic 1. Matrices and linear systems of equations.

(1) Special matrices: echelon form, reduced echelon form (Lecture 2), identity matrix, zero matrix.
(2) Algorithm to solve linear systems $A\mathbf{x} = \mathbf{b}$: apply Gauss-Jordan elimination of the matrix $(A|\mathbf{b})$ and write the *general form* of a solution using independent variables. (Lectures 2 and 3)
(3) Three scenarios: no solutions, exactly one solution or infinitely many solutions (number of free parameters = number of independent variables = number of columns of A - rank of the matrix A). (Lecture 4)
(4) Homogeneous vs inhomogeneous systems: $\mathbf{b} = \mathbf{0}$ or not. Homogeneous systems always have solutions, e.g. the trivial solution $x_1 = \ldots = x_n = 0$. (Lecture 5)
(5) Operations on matrices and their algebraic properties: addition, scalar multiplication, multiplication of matrices (for AB, need number of columns of A = number of rows of B), transposition. (Lectures 6 and 7)
(6) Computing inverses of square matrices: start from $(A|Id)$ and do Gauss-Jordan elimination to get $(A'|B)$. Then A is invertible if and only if $A' = Id$. Moreover, in that situation, $A^{-1} = B$. Special formula for 2×2 matrices (Lecture 8).
(7) Algebraic Properties of inverses: inverses of products and transposes (Lecture 8)
(8) Alternative characterization of invertible square matrices: (i) $det(A) \neq 0$; (ii) columns are linearly independent; (iii) all systems $A\mathbf{x} = \mathbf{b}$ have unique solutions; (iv) A is non-singular (its NullSpace is $\{\mathbf{0}\}$). ((i) is in Lecture 29 §5, the rest is in Lecture 9)

Topic 2. Geometry of vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$, lines and planes, the dot and cross products

(1) Magnitude and direction of vectors, addition and scalar multiplication of vectors; Triangle and Parallelogram Laws for addition (Lecture 10)
(2) Algebraic Properties of dot product; $|\mathbf{u}\cdot\mathbf{v}| = |\mathbf{u}||\mathbf{v}|\cos(\theta)$; the projection of a vector $\mathbf{u}$ along another non-zero vector $\mathbf{v}$ equals $\text{proj}_{\mathbf{v}}\,\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\mathbf{v}$. (Lecture 11)
(3) Determinantal formula for cross product in $\mathbb{R}^3$; $|\mathbf{u}\times\mathbf{v}| = |\mathbf{u}||\mathbf{v}|\sin(\theta)$; Algebraic properties of the cross products, $\mathbf{u}\times\mathbf{v} \perp \mathbf{u}$ and $\mathbf{u}\times\mathbf{v} \perp \mathbf{v}$. (Lectures 11-12)
(4) Parametric and vector equations for lines (and segments) in $\mathbb{R}^2$ and $\mathbb{R}^3$ (Lecture 12)
(5) The equation $\eta\cdot(x-x_0, y-y_0, z-z_0) = 0$ for planes in $\mathbb{R}^3$ using a point $P = (x_0, y_0, z_0)$ in the plane and the normal direction $\eta = \mathbf{v}\times\mathbf{w}$, where $\mathbf{v}, \mathbf{w}$ are the independent directions defining the plane. Formula to determine when four points P, Q, R, S in $\mathbb{R}^3$ are coplanar: $|(P - Q)\cdot((R - Q)\times(S - Q)| = 0$. (Lecture 13)

Topic 3. The vector space $\mathbb{R}^n$: subspaces, spanning sets, linear independence, bases of subspaces

(1) 10 properties characterizing the vector space $\mathbb{R}^4$, including the role of the zero vector $\mathbf{0}$ as a neutral element for addition, and $-\mathbf{v} = (-1)\mathbf{v}$ as an additive inverse for $\mathbf{v}$. (Lecture 14)
(2) 3 properties for determining subspaces of $\mathbb{R}^4$. Main examples: $\{\mathbf{0}\}$, $\mathbb{R}^n$, spanning of a finite number of vectors, solutions to *homogeneous* systems of equations in n variables, NullSpace of a matrix, Row Space of a matrix and Range = Column Space of a matrix. (Lecture 14-15)
(3) Definition of Linear independence and a spanning set (Lecture 15)
(4) Row space of a matrix is preserved under row operations, but NOT the column space. (Lecture 15)
(5) Definition of basis for a subspace $\mathbb{V}$ of $\mathbb{R}^n$; 2 algorithms to build a basis of a subspace from a spanning set; coordinates of a vector in $\mathbb{V}$ with respect to a fixed basis for $\mathbb{V}$. (Lecture 15-16)
(6) The dimension of a subspace $\mathbb{V}$ as the size of any basis for it. Important consequences: (i) a spanning set of size $\dim(\mathbb{V})$ is automatically a basis for $\mathbf{V}$; (ii) a linearly independent set inside $\mathbf{V}$ of size $\dim(\mathbb{V})$ is automatically a basis for $\mathbf{V}$; (iii) a subspace $\mathbf{V}$ of $\mathbb{R}^n$ of dimension n must equal $\mathbb{R}^n$; (iv) two subspaces of the same dimension with one contained in the other are automatically equal. (Lecture 17)
(7) Definition of the nullity and rank of an $m \times n$ matrix; Property: $\text{rank}(A) = \text{rank}(A^T)$; The rank-nullity theorem: $\text{rank}(A) + \text{nullity}(A) = n$. (Lecture 17).
(8) Orthogonal basis for subspaces of $\mathbb{R}^n$ via Gram-Schmidt method. For orthonormal basis: adjust the output of GS by dividing each vector by its magnitude (Lecture 18)

Topic 4. Abstract vector spaces: subspaces, spanning sets, linear independence, basis, coordinates with respect to a basis

(1) 10 properties characterizing an abtract vector space $\mathbf{W}$ with addition and scalar multiplication, including the role of the zero vector $\mathbf{0}$ as the *unique* neutral element for addition, and $-\mathbf{v} = (-1)\mathbf{v}$ as the *unique* additive inverse for $\mathbf{v}$. (Lecture 19)
(2) Main examples: $\mathbb{R}^n$, the space $\text{Mat}_{n\times m}$ of $n \times m$ matrices, the space $\mathcal{P}_n$ of polynomials of degree $\leq n$, the space $C_{[a,b]}$ of continuous functions on $[a, b]$. (Lecture 19)
(3) 3 properties for determining subspaces of $\mathbf{W}$. Main Examples: sets of vectors in the spaces from (2) subject to homogeneous linear constraints; spanning of a finite number of vectors in $\mathbf{W}$. (Lecture 20)
(4) Linear independence for vectors in $\mathbf{W}$ and methods to determine l.i. from a dependency relation: (i) for matrices, solve a linear system (one for each entry of the matrix computed from the relation); (ii) for polynomials or functions, evaluate f (and derivatives) at various x to obtain linear constraints of the scalars. (Lecture 21)
(5) Definition of basis for $\mathbf{W}$; algorithm to build basis from a spanning set (use linear dependencies to remove redundant vectors). Favorite examples: $E = \{e_1, \ldots, e_n\}$ for $\mathbb{R}^n$, $\{E_{11}, E_{12}, \ldots, E_{mn}\}$ for $\text{Mat}_{m\times n}$, $\{1, x, x^2, \ldots, x^n\}$ for $\mathcal{P}_n$ (Lecture 21)
(6) Use coordinates $[\_]_B$ with respect to a fixed basis B for $\mathbb{W}$ to fast determine if a set is a spanning set/l.i./basis: just check the statement for the coordinates of the vectors and use the standard tricks from $\mathbb{R}^n$ where $n = \dim(\mathbb{W})$. Using this we have the same 4 consequences from (6) in Topic 3. (Lecture 22)

Topic 5. Linear transformations between (abstract) vector spaces

(1) Definition: a map $T\colon \mathbb{R}^n \to \mathbb{R}^m$ that interacts well with addition and scalar multiplication. So it satisfies (i) $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u})+T(\mathbf{v})$, and (ii) $T(\alpha\mathbf{v}) = \alpha T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v}$ and α. Linear transformations are *completely* determined by assigning any vectors $\mathbf{w}_1, \ldots, \mathbf{w}_n$ in $\mathbb{R}^m$ to a *basis* $B = \{\mathbf{v}, \ldots, \mathbf{v}_n\}$ of $\mathbb{R}^n$. If B is the standard basis of $\mathbb{R}^n$ then $T(\mathbf{v}) = A\mathbf{v}$ where A is the $m \times n$ matrix $A = [T(e_1), \ldots T(e_n)]$. (Lecture 23)

(2) Matrix representations of linear transformations $T\colon \mathbb{R}^n \to \mathbb{R}^m$: pick a basis $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ and write $[T]_B = [T(\mathbf{v}_1), \ldots, T(\mathbf{v}_n)]$. Then: $T(\mathbf{v}) = [T]_B[\mathbf{v}]_B$ (just multiply the matrix $[T]_B$ by the coordinates of $\mathbf{v}$ with respect to the basis B.) Special case: if we pick standard basis $E = \{e_1, \ldots, e_n\}$ for $\mathbb{R}^n$, then $[T]_E$ is the matrix A from (1). (Lecture 24)

(3) The Null Space and Range of T can be computed from any matrix representation $[T]_B$; Rank-Nullity theorem for T: $\dim(\mathrm{NullSpace}(T)) + \dim(\mathrm{Range}(T)) = n$, and it agrees with the rank-nullity theorem for the matrix $[T]_B$. (Lecture 23).

(4) Linear transformations between abstract vector spaces $\mathbb{V}$ and $\mathbb{V}'$: choosing coordinates with respect to two basis (B for $\mathbb{V}$ and B' for $\mathbb{V}'$) identifies $T\colon \mathbb{V} \to \mathbb{V}'$ with a linear transformation $\tilde{T}\colon \mathbb{R}^{\dim(\mathbb{V})} \to \mathbb{R}^{\dim(\mathbb{V}')}$. More precisely: $[T(\mathbf{v})]_{B'} = \tilde{T}([\mathbf{v}]_B)$. Matrix representation for T: $[T]_{B,B'} = [[T(\mathbf{v_1})]_{B'}, \ldots, [T(\mathbf{v_1})]_{B'}]$, i.e. columns are coordinates w.r.t. B' for the image of each vector $\mathbf{v}_i$ in B (Lectures 24 and 27)

(5) A linear transformation $T\colon \mathbb{V} \to \mathbb{V}'$ is:
  (i) injective if and only if $\mathrm{NullSpace}(T) = \{\mathbf{0}_{\mathbb{V}}\}$.
  (ii) surjective if and only if $\mathrm{Range}(T) = \mathrm{Sp}(B) = \mathbb{V}'$ for any basis B for $\mathbb{V}$ (enough to check they have the same dimension; we get $\dim(\mathrm{Range}(T))$ from the Rank-Nullity Theorem on (3)).
  (iii) invertible (meaning injective and surjective) if $[T]_{B,B'}$ is invertible for any choice of bases B for $\mathbb{V}$ and B' for $\mathbb{V}'$. If so, $[T^{-1}]_{B',B} = ([T]_{B,B'})^{-1}$.

  Rank Nullity theorem for T says $\dim(\mathrm{NullSpace}(T)) + \dim(\mathrm{Range}(T)) = \dim(\mathbb{V}')$. (Lecture 25 and 26)

(6) Operations on linear transformations: addition, scalar multiplication, compositions. This can all be seen on the matrix representations (table on page 1 of Lecture 26, and pages 6-8 of Lecture 27). In particular, for compositions of $F\colon \mathbb{V} \to \mathbb{V}'$ and $G\colon \mathbb{V}' \to \mathbb{W}$ we get $[G \circ F]_{B,B'} = [G]_{B',B''}[F]_{B,B'}$ for any choice of bases B, B' and B'' for $\mathbb{V}, \mathbb{V}'$ and $\mathbb{W}$.
  Properties: $\mathrm{NullSpace}(F) \subseteq \mathrm{NullSpace}(G \circ F)$ and $\mathrm{Range}(G \circ F) \subseteq \mathrm{Range}(G)$. (Lecture 26)

(7) Many properties and examples of matrix representations are written in Lecture 27.

Topic 6. Determinants of square matrices

(1) Recursive definition, starting with 2×2 determinants and using cofactor expansion; cofactor expansion can be done along any row or column of the input matrix. Special cases: if lower or upper triangular, the determinant is the product of the diagonal entries. (Lecture 28);

(2) Effect of elementary operations on matrices (table on page 1 of Lecture 19); Consequence: a square matrix is invertible if and only if its determinant is nonzero. (Lecture 19).

(3) Algebraic properties of determinants: (i) $\det(A) = \det(A^T)$, (ii) $\det(A^{-1}) = 1/det(A)$, (iii) $\det(A) = \det(A^T)$, and (iv) $\det(AB) = \det(A)\det(B)$. (Lecture 30)
(4) Cramer's Rule for solving square linear systems of equations with invertible coefficient matrix via determinants. (Lecture 30)

Topic 7. Eigenvalues, eigenvectors and diagonalization of square matrices

(1) The Characteristic polynomial of A is $P_A(t) = \det(A - t\,Id)$. Definition of real eigenvalues of A: (i) real roots of $P_A(t)$; (ii) values λ in $\mathbb{R}$ where nullity$(A - \lambda Id) \neq 0$. Properties: (i) eigenvalues interact nicely with powers of A, inverses and translation $A + \mu Id$, (ii) $P_A(t) = P_{A^T}(t)$ for any matrix (Lectures 31—33)
(2) Eigenspaces $E_\lambda = \text{NullSpace}(A - \lambda Id)$; geometric multiplicity of $\lambda = \dim(E_\lambda) \leq$algebraic multiplicity of λ as a root of $P_A(t)$.
Defective matrices: for some eigenvalue λ, its geometric multiplicity is strictly smaller than its algebraic multiplicity. (Lecture 33)
(3) Diagonalization of an $n \times n$ matrix A over $\mathbb{R}$: find a basis for $\mathbb{R}^n$ consisting entirely of eigenvectors of A. Special case: A has n distinct real eigenvectors. (Lecture 33)
(4) Complex numbers $a + \mathbf{i}\,b$ for a, b in $\mathbb{R}$ with $\mathbf{i}^2 = -1$. Addition, multiplication, complex conjugation, modulus of a complex number; algebraic properties of these operations; $z^{-1} = \bar{z}/|z|^2$; $|zw| = |z||w|$; Fundamental theorem of algebra. The roots of a polynomial in $\mathbb{R}[x]$ real or they come in conjugate pairs $(\lambda, \overline{\lambda})$. (Lecture 34)
(5) The vector space $\mathbb{C}^n$ (use scalars in $\mathbb{C}$ rather than $\mathbb{R}$, the rest is the same as $\mathbb{R}^n$); Complex eigenvalues: complex roots of the characteristic polynomial $P_A(t)$. Complex eigenvectors: NullSpace of $A - \lambda Id$. We will have a basis for E_λ of size $\leq$ algebraic multiplicity of λ as a root of $P_A(t)$. The difficulty in working with $\mathbb{C}^n$ vs. $\mathbb{R}^n$ is computational, not conceptual. (Lecture 35)
(6) Key property: if A has real entries and λ is a complex eigenvalue of A, then $\dim E_\lambda = \dim E_{\overline{\lambda}}$. Moreover, $B = \{v_1, \ldots, v_p\}$ is a basis for E_λ then $\bar{B} = \{\overline{v_1}, \ldots, \overline{v_p}\}$ is a basis for $E_{\overline{\lambda}}$. (Lecture 35)
(7) Similarity of matrices: $A \simeq C$ if $C = S^{-1}AS$ for an invertible matrix S. Properties: same characteristic polynomial, same eigenvalues (with the same algebraic multiplicity), same determinant. Key property if $C = S^{-1}AS$: S defines an *invertible* linear transformation $T : E_\lambda(C) \to E_\lambda(A)$, with $T(\mathbf{v}) = S\mathbf{v}$. Similar matrices represent the same linear transformation of $\mathbb{R}^n$. (Lecture 36)
(8) Diagonalization: A is similar to a diagonal matrix C with eigenvalues along the diagonal. The columns of S must be an ordered basis of eigenvectors of A (using the order given by the diagonal entries of C). This can be done over $\mathbb{R}$ or over $\mathbb{C}$. Advantages: C^k diagonalizes A^k, C^{-1} diagonalizes A^{-1}, $C + \mu Id$ diagonalizes $A + \mu Id$ (all with the same matrix S). It's easier to compute C^k, C^{-1} than A^k and A^{-1}. (Lecture 36).
(9) Diagonalization is not always possible, even if we allow complex eigenvalues. Special case: real symmetric matrices are ALWAYS diagonalizable and their eigenvalues are real. We can find an orthogonal basis of eigenvectors, so $S^{-1} = S^T$ for such a choice. (Lecture 35-36)