

Example 1: 
$$\begin{cases} X_2 - X_3 + X_4 - X_5 = 1 \\ X_1 - 3X_2 + X_3 - X_4 + X_5 = 3 \\ -2X_2 + 2X_3 + X_4 - X_5 = 2 \\ X_2 - X_3 + 7X_4 - 7X_5 = 9 \end{cases} \rightarrow \left[ \begin{array}{ccccc|c} 0 & 1 & -1 & 1 & -1 & 1 \\ \textcircled{1} & -3 & 1 & -1 & 1 & 3 \\ 0 & -2 & 2 & 1 & -1 & 2 \\ 0 & 1 & -1 & 7 & -7 & 9 \end{array} \right] = B$$

Solve: 
$$\xrightarrow{R_1 \rightarrow R_2 \text{ (STEP 2)}} \left[ \begin{array}{ccccc|c} 1 & -3 & 1 & -1 & 1 & 3 \\ 0 & \textcircled{1} & -1 & 1 & -1 & 1 \\ 0 & -2 & 2 & 1 & -1 & 2 \\ 0 & 1 & -1 & 7 & -7 & 9 \end{array} \right] \xrightarrow{R_3 \rightarrow R_3 - (-2)R_2 \text{ (STEP 4)}} \left[ \begin{array}{ccccc|c} 1 & -3 & 1 & -1 & 1 & 3 \\ 0 & \textcircled{1} & -1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 3 & -3 & 4 \\ 0 & \textcircled{1} & -1 & 7 & -7 & 9 \end{array} \right]$$

$$\xrightarrow{R_4 \rightarrow R_4 - R_2 \text{ (STEP 4)}} \left[ \begin{array}{ccccc|c} 1 & -3 & 1 & -1 & 1 & 3 \\ 0 & 1 & -1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 3 & -3 & 4 \\ 0 & 0 & 0 & 6 & -6 & 8 \end{array} \right] \xrightarrow{R_3 \rightarrow \frac{1}{3}R_3 \text{ (STEP 2)}} \left[ \begin{array}{ccccc|c} 1 & -3 & 1 & -1 & 1 & 3 \\ 0 & 1 & -1 & 1 & -1 & 1 \\ 0 & 0 & 0 & \textcircled{1} & -1 & \frac{4}{3} \\ 0 & 0 & 0 & \textcircled{6} & -6 & 8 \end{array} \right]$$

$$\xrightarrow{R_4 \rightarrow R_4 - 6R_3 \text{ (STEP 4)}} \left[ \begin{array}{ccccc|c} 1 & -3 & 1 & -1 & 1 & 3 \\ 0 & 1 & -1 & 1 & -1 & 1 \\ 0 & 0 & 0 & \textcircled{1} & -1 & \frac{4}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

From EF to REF: only need to treat  $R_3$  &  $R_2$   
 column 4 & column 2 independent variables

$$\xrightarrow{R_2 \rightarrow R_2 - R_3} \left[ \begin{array}{ccccc|c} 1 & -3 & 1 & 0 & 0 & \frac{13}{3} \\ 0 & 1 & -1 & 0 & 0 & -\frac{1}{3} \\ 0 & 0 & 0 & \textcircled{1} & -1 & \frac{4}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right] \xrightarrow{R_1 \rightarrow R_1 - (-3)R_2 \text{ (STEP 6 for col 2)}} \left[ \begin{array}{ccccc|c} \textcircled{1} & 0 & -2 & 0 & 0 & \frac{10}{3} \\ 0 & \textcircled{1} & -1 & 0 & 0 & -\frac{1}{3} \\ 0 & 0 & 0 & \textcircled{1} & -1 & \frac{4}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

{ indep. variables =  $X_3, X_5$  (complement)  
 { dependent variables =  $X_1, X_2, X_4$  (① variables)

Set of Solutions: 
$$\begin{cases} X_1 - 2X_3 = \frac{10}{3} \\ X_2 - X_3 = -\frac{1}{3} \\ X_4 - X_5 = \frac{4}{3} \end{cases} \text{ equivalently } \begin{cases} X_1 = \frac{10}{3} + 2X_3 \\ X_2 = -\frac{1}{3} + X_3 \\ X_4 = \frac{4}{3} + X_5 \end{cases} \text{ for any } X_3, X_5$$

So General Solution has the form  $(\frac{10}{3} + 2X_3, -\frac{1}{3} + X_3, X_3, \frac{4}{3} + X_5, X_5)$

Particular solutions:  $\therefore X_3 = X_5 = 0 \rightsquigarrow (\frac{10}{3}, -\frac{1}{3}, 0, \frac{4}{3}, 0)$   
 (examples)

$\cdot X_3 = 1, X_5 = -1 \rightsquigarrow (\frac{16}{3}, -\frac{4}{3}, 1, \frac{1}{3}, -1)$

Example 2 Solve 
$$\begin{cases} 2x_1 + 3x_2 - 4x_3 = 3 \\ x_1 - 2x_2 - 2x_3 = -2 \\ -x_1 + 16x_2 + 2x_3 = 16 \end{cases}$$

Augmented matrix 
$$\left[ \begin{array}{ccc|c} 2 & 3 & -4 & 3 \\ 1 & -2 & -2 & -2 \\ -1 & 16 & 2 & 16 \end{array} \right]$$

Scale step  
 $R_1 \rightarrow \frac{1}{2}R_1$

Replacement Step  
 $R_2 \rightarrow R_2 - R_1$   
 $R_3 \rightarrow R_3 + R_1$

first non-zero column & non-zero entry in row 1  
Rescale  
 $R_2 \rightarrow \frac{-2}{7}R_2$

$R_3 \rightarrow R_3 - \frac{35}{2}R_2$

$$\left[ \begin{array}{ccc|c} 1 & \frac{3}{2} & -2 & \frac{3}{2} \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

REF  $\rightarrow$  REF step.  
 $R_1 \rightarrow R_1 - \frac{3}{2}R_2$

Equiv System: 
$$\begin{cases} x_1 - 2x_3 = 0 \\ x_2 = 1 \end{cases}$$

general form 
$$\begin{cases} x_1 = 2x_3 \\ x_2 = 1 \end{cases}$$

$(x_1, x_2, x_3) = (2x_3, 1, x_3) = (0, 1, 0) + x_3(2, 0, 1)$  infinitely many solutions

Example 3: Solve 
$$\begin{cases} 2x_1 + 3x_2 - 4x_3 = 3 \\ x_1 - 2x_2 - 2x_3 = -2 \\ -x_1 + 16x_2 + 2x_3 = 16 + \frac{1}{2} \end{cases}$$

B = augmented matrix is

row equivalent to

$$\left[ \begin{array}{ccc|c} 1 & \frac{3}{2} & -2 & \frac{3}{2} \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 + \frac{1}{2} \end{array} \right]$$

REF matrix 
$$\left[ \begin{array}{ccc|c} 1 & \frac{3}{2} & -2 & \frac{3}{2} \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & \frac{1}{2} \end{array} \right]$$

(until \*) we do the same row elementary operations as in Example 2)

INCOMPATIBLE system!  
NO solutions!

Example 4: Solve 
$$\begin{cases} 2x_1 + 3x_2 - 4x_3 = 3 \\ x_1 - 2x_2 - 2x_3 = -2 \\ -x_1 + 16x_2 + 3x_3 = 16 \end{cases}$$

First two step as in Example 2

System gives augmented matrix row equivalent to

$R_3 \rightarrow R_3 - \frac{35}{2}R_2$

$$\left[ \begin{array}{ccc|c} 1 & \frac{3}{2} & -2 & \frac{3}{2} \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array} \right]$$

$R_1 \rightarrow R_1 + 2R_3$

$R_1 \rightarrow R_1 - \frac{3}{2}R_2$

system 
$$\begin{cases} x_1 = 0 \\ x_2 = 1 \\ x_3 = 0 \end{cases}$$

Unique Solution =  $(0, 1, 0)$ .

REF matrix