Lecture II: §1.1 Matrices, §1.2 Echelon forms

§1 Matrices & linear systems

Recall: An $\underline{\underline{m}} \times n$ system of linear equations is a set of $\underline{\underline{m}}$ linear equations in n unknowns. We write:

$$(*) \begin{cases} a_{11} \boxed{x_1} + a_{12} \boxed{x_2} + \cdots + a_{1n} \boxed{x_n} = b_1 & (\text{Eqn } 1) \\ a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n = b_2 & (\text{Eqn } 2) \\ \vdots & \vdots \\ a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n = b_m & (\text{Eqn } m) \end{cases}$$

where,

- $a_{11} \ldots a_{mn}$ are coefficients (in $\mathbb{Q}, \mathbb{R}, \mathbb{C}, \ldots$) ($m \cdot n$ many $\rightsquigarrow (a_{ij})_{\substack{1 \le i \le m \\ 1 \le j \le n}}$) i = eqn number, j = unknown index.
- m constant terms $b_1, \ldots, b_m$ (in $\mathbb{Q}, \mathbb{R}, \mathbb{C}, \ldots$) $\rightsquigarrow (b_i)_{1 \le i \le m}$
- n unknowns $x_1, \ldots, x_n$

Collect the coefficients & constant terms into matrices. This gives a convenient framework to represent & solve linear systems.

Def: An $\underline{\underline{m}} \times n$ matrix A is a rectangular array of numbers with m rows & n columns. Write them as:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = (a_{ij})_{i,j}$$

row # column #

Example: $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 0 \end{bmatrix}$ 2×3 matrix. $A = \begin{bmatrix} 1 & 0 \\ 6 & 1 \end{bmatrix}$ 2×2 matrix

- If $n = m$ we say A is a square matrix.

§2 Representing linear systems via matrices:
- equations $\rightsquigarrow$ rows
- unknowns $\rightsquigarrow$ columns & constant terms

Def: • The coefficient matrix for the system $(*)$ is the $m \times n$ matrix $A = (a_{ij})$

• The augmented matrix ——————— $m \times (n+1)$ — B, where $B = [A \mid \underline{b}] = \begin{array}{c} \\ \left[\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ a_{21} & \cdots & a_{2n} & b_2 \\ \vdots & & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} & b_m \end{array}\right] \\ \text{(col 1 for } x_1) \quad \text{(col } n \text{ for } x_n) \quad \text{(col } n+1 \text{ for } \underline{b}_i\text{'s)} \end{array} \begin{array}{l} \leftarrow \text{row 1} \\ \leftarrow \text{row 2} \\ \\ \leftarrow \text{row } m \end{array}$ where $\underline{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$

(2)

Example: $\begin{cases} x_1 + 3x_2 - 2x_3 = 4 \\ 4x_1 \quad - 5x_3 = 0 \\ -2x_1 + 6x_2 \quad = 24 \end{cases} \rightsquigarrow_{\text{Align}}$ $\begin{array}{l} \underset{\llcorner\lrcorner}{1}\, x_1 + \underset{\llcorner\lrcorner}{3}\, x_2 + \underbrace{(-2)}\, x_3 = 4 \\ \underset{\llcorner\lrcorner}{4}\, x_1 + \underset{\llcorner\lrcorner}{0} \cdot x_2 + \underbrace{(-5)}\, x_3 = 0 \\ \underbrace{(-2)}\, x_2 + \underset{\llcorner\lrcorner}{6}\, x_2 + \underset{\llcorner\lrcorner}{0} \cdot x_3 = 24 \end{array}$

"missing terms" = "coeffs equal 0"

$\rightsquigarrow B = \left[\begin{array}{ccc|c} 1 & 3 & -2 & 4 \\ 4 & 0 & -5 & 0 \\ -2 & 6 & 0 & 24 \end{array}\right]$

Last time: manipulated (combined multiplication by scalars with adding/substracting 2 equations) to solve systems.

Upshot = can use certain allowed operations on the system to go from one system to a <u>simpler one</u> <u>without changing the solution set</u>. Have parallel operations on the assoc. augmented matrix side.

§2. Elementary operations

2 steps to solve a system: (1) Reduce to a simpler equivalent one (eliminating variables)

(2) Describe the solutions from the simpler one

Def: 2 systems in $\underline{n}$ unknowns are <u>equivalent</u> if they have the <u>same solution set</u>

Examples: (1) $\begin{cases} x + y = 5 \\ 2x + 2y = 10 \end{cases}$ & $\{x + y = 5$ are equivalent

(2) $\begin{cases} 2x + 4y = 18 \\ 4x - 4y = 0 \end{cases}$ (last time) & $\begin{cases} x = 3 \\ y = 3 \end{cases}$ are equivalent.

(3) $\begin{cases} x + y = 0 \\ x + y = 1 \end{cases}$ & $\{0 = 1$ are equivalent (they both have <u>no</u> solutions).

Theorem 1: The following <u>elementary operations</u> give equivalent systems:

(1) interchange 2 equations (E_i & E_j, write $E_i \longleftrightarrow E_j$)

(2) multiply an equation by a <u>nonzero number</u> & replace it (E_i, $\alpha \neq 0$, write $E_i \longrightarrow \alpha E_i$)

(3) replace an equation by adding it a constant multiple of a different equation ($E_i \longrightarrow E_i + \lambda E_j$ for $j \neq i$ & <u>any</u> fixed number λ)

Why does this work? (1) is clear

(2) Since $\alpha \neq 0$, we can reverse the operation $E_i \to \alpha E_i \to \frac{1}{\alpha}(\alpha E_i) = E_i$ ($\frac{1}{\alpha}$: $\neq 0$)

(3) Can "reverse" this operation as well:

$$\begin{cases} E_i \\ \text{rest} \end{cases} \longrightarrow \begin{cases} (E_i + \lambda E_j) = \text{new } E_i' \\ \text{rest} \end{cases} \longrightarrow \begin{cases} (E_i + \lambda E_j) + (-\lambda) E_j = E_i \\ \text{rest.} \end{cases}$$

<u>Example</u>:

$$\begin{cases} x_1 + x_2 = 3 \\ -x_1 + 3x_2 = 9 \end{cases} \xrightarrow[E_1 \to E_1 + E_2]{} \begin{cases} 4x_2 = 12 \\ -x_1 + 3x_2 = 9 \end{cases} \xrightarrow[E_1 \to \frac{1}{4} E_1]{} \begin{cases} x_2 = 12/4 = 3 \\ -x_1 + 3x_2 = 9 \end{cases}$$

$$\xrightarrow[E_2 \to E_2 - 3E_1]{} \begin{cases} x_2 = 3 \\ -x_1 = 9 - 9 = 0 \end{cases} \xrightarrow[E_2 \to -E_2]{} \boxed{\begin{cases} x_2 = 3 \\ x_1 = 0 \end{cases}}$$

EASY TO SOLVE! (unique solution)

By Thm 1, we conclude $\begin{cases} x_1 + x_2 = 3 \\ -x_1 + 3x_2 = 9 \end{cases}$ has a unique soln $= (0, 3)$.

§ 4. On the matrix side = Row operations

Def: There are 3 elementary row operations corresponding the elementary operat. on systems:

(1) interchanging two rows ($R_i \leftrightarrow R_j$)

(2) replace a row by a non-zero scalar multiple of it ($R_i \to \alpha R_i$ for $\alpha \neq 0$)

(3) replace a row by adding to it a constant multiple of a different row. ($R_i \to R_i + \lambda R_j$ for $i \neq j$, any λ)

Def: Two matrices are <u>row equivalent</u> if we can obtain one from the other by a sequence of elementary row operations.

<u>Example</u>: (above)

$$\left[\begin{array}{cc|c} 1 & 1 & 3 \\ -1 & 3 & 9 \end{array}\right] \to \left[\begin{array}{cc|c} 0 & 4 & 12 \\ -1 & 3 & 9 \end{array}\right] \to \left[\begin{array}{cc|c} 0 & 1 & 3 \\ -1 & 3 & 9 \end{array}\right] \to \left[\begin{array}{cc|c} 0 & 1 & 3 \\ -1 & 0 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 0 & 1 & 3 \\ 1 & 0 & 0 \end{array}\right]$$

$$\xrightarrow[R_1 \leftrightarrow R_2]{} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 3 \end{array}\right]$$

reduced echelon form

Obs: If two row equivalent matrices record two linear systems (as their augmented matrices), then these systems are equivalent.

$\rightsquigarrow$ Algorithm for solving $m \times n$ linear systems:

ALGORITHM: (1) Write the augmented matrix B of the input system

(2) do elementary row operations to go from B to a simpler matrix B' (Gauss-Jordan Elimination)

(3) solve the system represented by B'

Ex above:

Step 1: $\begin{cases} x + y = 3 \\ -x + 3y = 9 \end{cases} \rightsquigarrow B = \left[\begin{array}{cc|c} 1 & 1 & 3 \\ -1 & 3 & 9 \end{array}\right]$

Step 2: From B we get $B' = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 3 \end{array}\right]$

Step 3: We write the system $\begin{cases} x = 0 \\ y = 3 \end{cases}$. It's already solved!

So the original system has a unique solution $(x, y) = (0, 3)$.

What do we mean by a simpler matrix?

- Staircase shape, where each row starts with a 1 and has 0's to the left & below each one of these 1's = Echelon form
- Reduced echelon form = 0's also above the 1's

Examples: $\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 3 \end{array}\right]$ RED ECH., $\left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 1 & 3 \end{array}\right]$ ECH.

$\left[\begin{array}{cccc|c} 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 3 \end{array}\right]$ ECH., $\left[\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$ ECH, $\left[\begin{array}{ccc|c} 0 & 0 & 1 & 1 \\ 1 & 0 & \boxed{2} & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$ not ECH

Next time: More on echelon & reduced echelon forms & Gauss-Jordan Elimination