Lecture V: §1.3 (cont) Consistent systems, §1.5 Matrix Operations

§1. Consistent vs Inconsistent:

Last Time: • Test for consistent vs inconsistent systems via $\underset{m\times(n+1)}{B'} = [A'|b']$ in REF

(Consistent = B' has no $[0 \dots 0 | 1]$ row.) $\boxed{\text{Rank}(B') = \#\text{ non-zero rows}}$

Theorem: • In the consistent case: number of free parameters $= n - \text{rank}(B')$

($\text{rank}(B') \leq \min\{m, n+1\}$

$\text{rank}(B') \leq \min\{m, n\}$ for consistent syst

In particular: unique solution if and only if (1) consistent

(2) $\text{rank}(B') = n$

Consequence: A linear system has $\underbrace{\text{one, infinitely many}}_{\text{consistent}}$ or $\underbrace{\text{no solutions}}_{\text{inconsistent}}$

Special case: If $m<n$ (more unknowns than equations), we have either no solution or infinitely many.

Why? $\text{rank}(B') \leq \min\{m, n\} = m < n$ for consistent systems, so $0 < n - \text{rank}(B')$ free parameters.

Examples:

① $B = \left[\begin{array}{ccc|c} 1 & 3/2 & -2 & 3/2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow[R_1 \to R_1 - \frac{3}{2}R_2 + 2R_3]{} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right] = B'$ consistent! REF

$\left.\begin{array}{l}\text{rank}(B') = 3 \\ n = 3\end{array}\right\}$ no free variables! Unique solution = $\boxed{\begin{array}{c} x_1 = 0 \\ x_2 = 1 \\ x_3 = 0 \end{array}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

② $B = \left[\begin{array}{ccc|c} 1 & 3/2 & -2 & 3/2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[R_1 \to R_1 - \frac{3}{2}R_2]{} \left[\begin{array}{ccc|c} 1 & 0 & -7/2 & 3/2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] = B'$ consistent!

dependent

$\left.\begin{array}{l}\text{rank}(B') = 2 \\ n = 3\end{array}\right\}$ 1 free parameter $= x_3$; x_1, x_2 = dependent vars

$\begin{cases} x_1 = \frac{3}{2} + \frac{7}{2}x_3 \\ x_2 = -x_3 \\ x_3 \text{ ANY} \end{cases}$ infinitely many solns!

Soln $(x_1, x_2, x_3) = (\frac{3}{2}, 0, 0) + x_3(\frac{7}{2}, -1, 1) = \begin{bmatrix} 3/2 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 7/2 \\ -1 \\ 1 \end{bmatrix}$

Parametric form of a line in $\mathbb{R}^3$

§2. Homogeneous vs Inhomogeneous systems

Def: A system with augmented matrix $B = [A|b]$ is homogeneous if all constant terms (b_i's) are zero. We say it is inhomogeneous, otherwise.

Why? • Gauss Jordan elimination preserves homogeneous systems, hence also inhomogeneous

• Homogeneous systems are ALWAYS CONSISTENT: $x_1 = \cdots = x_n = 0$ is always a soln. (trivial or zero solution)

$$\begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n = 0 \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = 0 \end{cases} \rightsquigarrow \text{We evaluate at } x_1 = x_2 = \cdots = x_n = 0 \text{ \& get } \begin{cases} 0 = 0 \\ \vdots \\ 0 = 0 \end{cases}$$

Thm: A homogeneous $m \times n$ system has infinitely many solutions if $m < n$

⚠ It will have them if $n > \operatorname{rank}(B')$, but this can also happen if $m \geq n$.

Example:
$$B = \left[\begin{array}{ccccc|c} 1 & 0 & -2 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \end{array}\right] \xrightarrow{R_2 \to R_2 - R_3} \left[\begin{array}{ccccc|c} 1 & 0 & -2 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \end{array}\right] = B' \quad \text{REF}$$

x_1, x_2, x_4 = dependent (rank $B' = 3$)

x_3, x_5 = indep.

$$\begin{cases} x_1 = 2x_3 \\ x_2 = x_3 - x_5 \\ x_4 = x_5 \\ x_3, x_5 \text{ ANY} \end{cases}$$

2 free parameters = dimension 2.

$$\text{Soln} = x_3(2, 1, 1, 0, 0) + x_5(0, -1, 0, 1, 1) = x_3\begin{bmatrix}2\\1\\1\\0\\0\end{bmatrix} + x_5\begin{bmatrix}0\\-1\\0\\1\\1\end{bmatrix}$$

(Plane in $\mathbb{R}^5$ through the origin $= (0,0,0,0,0)$)

Application: Find conics in $\mathbb{R}^2$ through a fixed set of points.

Def: A conic in $\mathbb{R}^2$ is
$$\boxed{ax^2 + bxy + cy^2 + dx + ey + f = 0} \qquad (*)$$
(degree 2 eqn in x, y)

for fixed values of a, b, c, d, e, f (coefficients), NOT all zero.

GOAL: Find coefficient values so that $(x_1, y_1), \ldots, (x_m, y_m)$ satisfy the equation $(*)$

• **How?** Evaluate at each (x_i, y_i) gives 1 linear eqn in $(a, b, \ldots, f)$

• Finding conic $=$ finding a non-trivial solution to this homog system of m equations in the 6 unknowns (a, b, c, d, e, f).

<u>Note</u>: If $m<6$ we will always have a nontrivial solution (at most 5 pts)

<u>Example</u>: Pick 5 points $(-1,0)$, $(0,1)$, $(2,2)$, $(2,-1)$, $(0,-3)$. & find the conic through them.

$P_1=(-1,0) \rightsquigarrow a(-1)^2+b(-1)\cdot 0+c\cdot 0^2+d(-1)+e\cdot 0+f=0$

becomes $a-b-d+f=0$

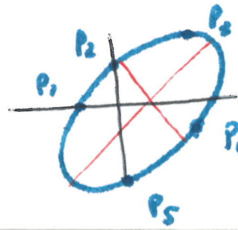
The system has augmented matrix

$$\begin{array}{cc} & \begin{array}{cccccc|c} a & b & c & d & e & f & \end{array} \\ B = & \left[\begin{array}{cccccc|c} 1 & -1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 4 & 4 & 4 & 2 & 2 & 1 & 0 \\ 4 & -2 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 9 & 0 & -3 & 1 & 0 \end{array}\right] \\ & \begin{array}{cccccc} x^2 & xy & y^2 & x & y & 1 \end{array} \end{array} \begin{array}{l} P_1=(-1,0) \\ P_2=(0,1) \\ P_3=(2,2) \\ P_4=(2,-1) \\ P_5=(0,-3) \end{array} \sim B' = \left[\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & 0 & 7/18 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1/2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & 1 & 0 & -11/18 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2/3 & 0 \end{array}\right]$$

<u>Soln</u>: $a=\frac{-7}{18}f$, $b=\frac{1}{2}f$, $c=\frac{-1}{3}f$, $d=\frac{11}{18}f$, $e=-\frac{2}{3}f$ $= f\begin{bmatrix} -7/18 \\ +1/2 \\ -1/3 \\ 11/18 \\ -2/3 \end{bmatrix} \rightsquigarrow$ <u>Eqn</u>: $f\left(\frac{-7}{18}x^2+\frac{1}{2}xy-\frac{1}{3}y^2+\frac{11}{18}x-\frac{2}{3}y+1\right)=0$

for any $f\neq 0$. Can take $f=18$, so we get

Conic: $\boxed{0=-7x^2+9xy-6y^2+11x-12y+18}$



(rotated ellipse)