Lecture VIII: § 1.9 (Matrix inverses & their properties)

Recall: A of size $n\times n$ is <u>invertible</u> if we can find a matrix B of size $n\times n$ satisfying $\boxed{AB = BA = I_n}$ (2 matrix identities!)

Example: $A = I_n = \begin{pmatrix} 1 & 0 & & 0 \\ 0 & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix}$ is invertible because $I_n \cdot I_n = I_n$.

Property: If B exists, it is unique. Call it A^{-1}.

Why? Imagine B & B' both satisfy $AB = BA = I_n = AB' = B'A$.

Then $B = BI_n = B(AB') \underset{\text{Assoc}}{=} (BA)B' = I_n B' = B'$.

So we conclude $B = B'$.

Q: Why we care about invertible matrices?

Proposition: If A has size $n\times n$ & is invertible then $A\underline{x} = \underline{b}$ in $\mathbb{R}^n$ is ALWAYS consistent (for any $\underline{b}$ in $\mathbb{R}^n$). Furthermore, $\underline{x} = A^{-1}\underline{b}$ is the unique solution.

Proof: To prove consistency, check the proposed $\underline{x} = A^{-1}\underline{b}$ is a solution.

$$A\underline{x} = A(A^{-1}\underline{b}) \underset{\text{Assoc}}{=} (AA^{-1})\underline{b} = I_n \cdot \underline{b} = \underline{b} \quad \checkmark$$

Now, we want to show this is the only solution. To do so, we multiply both sides of $=$ in the equation to the left by A^{-1}.

$A\underline{x} = \underline{b}$ for any solution $\underline{x}$ in $\mathbb{R}^n$

$$\text{Assoc}\begin{cases} A^{-1}(A\underline{x}) = A^{-1}\underline{b} \\ \underbrace{\underbrace{(A^{-1}A)}_{=I_n}\underline{x}}_{=\underline{x}} = A^{-1}b \end{cases}$$

$\rightsquigarrow$ We conclude $\underline{x} = A^{-1}b$ is the only solution.

Non-example: $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is non-invertible. ($\Delta = 1\cdot 4 - 2\cdot 3 = 4-6 = -2 \neq 0 \Rightarrow$ invertible (see page 4))

Why? Propose $B = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}$ & show $AB = I_2$ admits no solution for B.

$AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} = \begin{bmatrix} x_{11} & x_{12} \\ 0 & 0 \end{bmatrix} \overset{?}{=} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ has no solution!

§1. Algorithms for building inverses:

Q1: How to test if A of size $n\times n$ is invertible?

Q2: If A is invertible, how to build A^{-1}?

We focus on Q2 to answer Q1. Later on, we'll give a different answer (using determinants).

Proposal: Write $B = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ \vdots & \vdots & & \vdots \\ x_{n1} & x_{n2} & & x_{nn} \end{bmatrix}$ and try to solve $AB = I_n$.

This leads to n-systems of linear equations in $\mathbb{R}^n$:

$$A\begin{bmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{n1} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad A\begin{bmatrix} x_{12} \\ x_{22} \\ \vdots \\ x_{n2} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \ldots\ldots, \quad A\begin{bmatrix} x_{1n} \\ x_{2n} \\ \vdots \\ x_{nn} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

Rather than solving each system separately, we can solve them all together since they share the matrix of coefficients & Gauss-Jordan elimination uses A to decide what row operations to perform:

In short:

Algorithm: $\Big[A \,\Big|\, \underbrace{\begin{matrix}1\\0\\\vdots\\0\end{matrix} \,\Big|\, \begin{matrix}0\\1\\\vdots\\0\end{matrix} \,\Big|\, \cdots \,\Big|\, \begin{matrix}0\\\vdots\\0\\1\end{matrix}}_{I_n}\Big] \underset{\text{Gauss Jordan}}{\sim} \Big[A' \,\Big|\, B'\Big]$ in REF.

Outcomes: ① If A' has no nonzero rows, since A' has size $n\times n$ & it's in REF, we conclude $A' = \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 0 & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & & 0 & 1 \end{pmatrix} = I_n$.

We get $AB' = I_n$ so B' solves $AB = I_n$ for $B = B'$.

This *almost* shows A is invertible, because we still need to check $B'A = I_n$

② If A' has some zero rows, then we'll conclude A is not invertible. To this end, we'll need to check that B' has no nonzero rows.

If so, $[A' \mid B']$ produces n systems and at least one of which is inconsistent. (More on this next time, but Upshot: $I_n \underset{\text{row equiv}}{\sim} B'$ so $\operatorname{rank}(I_n) = \operatorname{rank} B' = n$ & so B' cannot have a row of zeros)

Q: How do we know B' output by the algorithm when $A' = I_n$ satisfies $B'A = I_n$?

A: We can reverse the Gauss Jordan elimination!

More precisely, we want to show $B'X = I_n$ has a unique solution, namely $\underset{\text{n×n matrix}}{X} = A$.

To this end, we run our algorithm from before:

$$(B' \mid I_n) \sim (B'' \mid C) \text{ in REF!}$$

But we know $(B'' \mid C)$ is unique, and $(A \mid I_n) \underset{\text{row equiv}}{\overset{\text{Gauss J.}}{\sim}} (I_n \mid B')$ (use $A' = I_n$)

So reverse the Gauss-Jordan and flip both sides of $\mid$ to get:

$$(B' \mid I_n) \sim (I_n \mid A)$$

By uniqueness of REF we get $(B'' \mid C) = (I_n \mid A)$ so $C = A$ is the unique solution to $B'X = I_n$.

We conclude: $AB' = B'A = I_n$ for Outcome ① of the Algorithm.

Example: $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 2 & 4 & 0 & 1 \end{array}\right] \xrightarrow[R_2 \to R_2 - 2R_1]{} \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & -2 & -2 & 1 \end{array}\right] \xrightarrow[R_2 \to -\frac{1}{2}R_2]{} \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 1 & 1 & -1/2 \end{array}\right]$$

$$\xrightarrow[R_1 \to R_1 - 3R_2]{} \left[\begin{array}{cc|cc} 1 & 0 & -2 & 3/2 \\ 0 & 1 & 1 & -1/2 \end{array}\right]$$

Conclusion: A is invertible & $A^{-1} = \begin{bmatrix} -2 & 3/2 \\ 1 & -1/2 \end{bmatrix}$

Check: $\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} -2 & 3/2 \\ 1 & 1/2 \end{bmatrix} = I_2$ ✓ (don't need to check the other product because of our earlier discussion.)

§2 The 2×2 case:

Write $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ & set $\Delta := ad - bc$ (determinant)

Rule: (1) If $\Delta = 0$, then A does not admit an inverse

(2) If $\Delta \neq 0$, ——— is invertible & $A^{-1} = \frac{1}{\Delta} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Proof: For (2): enough to check $A \frac{1}{\Delta} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} A = I_2$.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \frac{1}{\Delta}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{\Delta}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{\Delta}\begin{bmatrix} \underbrace{ad-bc}_{=\Delta} & 0 \\ 0 & \underbrace{-bc+ad}_{=\Delta} \end{bmatrix} = I_2$$

jumps (it's a scalar)

$$\frac{1}{\Delta}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{\Delta}\begin{bmatrix} \overbrace{ad-bc}^{=\Delta} & 0 \\ 0 & \underbrace{-bc+ad}_{=\Delta} \end{bmatrix} = I_2 .$$

TODO (1): We try to solve $AB = I_2$ for $B = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}$ & see that we will always fail if $\Delta = ad-bc = 0$.

CASE 1: If $b=0$ we get $ad=0$ so either $a=0$ or $d=0$.

- If $a=0$, then $A = \begin{bmatrix} 0 & 0 \\ c & d \end{bmatrix}$ & $A\begin{bmatrix} x_{11} \\ x_{21} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ has no solution! so A is not invertible
- If $a \neq 0$, then $d=0$ so $A = \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix}$

Run the algorithm for building A^{-1}:

$$\left[\begin{array}{cc|cc} a & 0 & 1 & 0 \\ c & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_1 \to \frac{1}{a} R_1]{} \left[\begin{array}{cc|cc} 1 & 0 & 1/a & 0 \\ c & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_2 \to R_2 - cR_1]{} \left[\begin{array}{cc|cc} 1 & 0 & 1/a & 0 \\ 0 & 0 & -c/a & 1 \end{array}\right]$$

allowed because $a \neq 0$

So the system $A\begin{bmatrix} x_{12} \\ x_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is inconsistent! A is not invertible.

CASE 2: Assume $b \neq 0$.

Then, use $ad-bc=0$ & solve for c: $c = \frac{ad}{b}$.

Again, run the algorithm for building A^{-1} using $A = \begin{bmatrix} a & b \\ \frac{ad}{b} & d \end{bmatrix}$

$$\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ \frac{ad}{b} & d & 0 & 1 \end{array}\right] \xrightarrow[R_2 \to R_2 - \frac{d}{b} R_1]{} \left[\begin{array}{cc|cc} a & b & 1 & 0 \\ 0 & 0 & -\frac{d}{b} & 1 \end{array}\right]$$

As before, we see from here that $A\begin{bmatrix} x_{12} \\ x_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is inconsistent!

And so A is not invertible.

In all 3 cases, we failed to build A^{-1}.

§2 Properties of inverses:

To finish, we write down how do inverse matrices interact with products, scalar multiplication & transpose:

Theorem 1: Fix A, C of size $n \times n$, both invertible. Fix $\alpha \neq 0$ scalar.

(1) A^{-1} is invertible & $(A^{-1})^{-1} = A$

(2) AC is ______ & $(AC)^{-1} = C^{-1}A^{-1}$

(3) αA is ______ & $(\alpha A)^{-1} = \frac{1}{\alpha} A^{-1}$

(4) A^T is ______ & $(A^T)^{-1} = (A^{-1})^T$

(5) I_n is ______ & $(I_n)^{-1} = I_n$.

Proof: All these properties can be proven by checking the (RHS) satisfy the equation for the inverse.

(1) $(A^{-1})A = A(A^{-1}) = I_2$ so A is the inverse of A^{-1}.

(2) $(AC)(C^{-1}A^{-1}) \overset{\text{Assoc}}{=} A\underbrace{(CC^{-1})}_{=I_n}A^{-1} = A I_n A^{-1} = AA^{-1} = I_n$

$(C^{-1}A^{-1})(AC) \overset{\text{Assoc}}{=} C^{-1}\underbrace{(A^{-1}A)}_{=I_n}C = C^{-1}I_nC = C^{-1}C = I_n$

(3) $(\alpha A)\left(\frac{1}{\alpha}A^{-1}\right) \overset{\text{scalars \& matrices commute}}{=} \alpha\left(\frac{1}{\alpha}AA^{-1}\right) = \underbrace{\left(\alpha \frac{1}{\alpha}\right)}_{=1}\underbrace{(AA^{-1})}_{=I_n} = I_n$

$\left(\frac{1}{\alpha}A^{-1}\right)(\alpha A) \overset{\text{scalars \& matrices commute}}{=} \left(\frac{1}{\alpha}\alpha\right)(A^{-1}A) = 1 I_n = I_n$.

(4) $(A^T)(A^{-1})^T \overset{\text{transpose rule}}{=} (A^{-1}A)^T = (I_n)^T = I_n$

$(A^{-1})^T A^T \overset{\text{transpose rule}}{=} (AA^{-1})^T = I_n^T = I_n$.

(5) $I_n I_n = I_n$ by definition so $I_n^{-1} = I_n$.

Question: Why does the algorithm for finding inverses work? It only guarantees $AB = I_n$. But why does $BA = I_n$ as well?

A: $(A \mid I_n) \xrightarrow[\text{Gauss Jordan}]{} (I_n \mid B)$ This says $I_n \underset{\text{row equiv}}{\sim} B$ & $AB = I_n$

& I_n is in REF so $\text{rank}(B) = \text{rank}(I_n) = n$

Now: $(B \mid I_n) \xrightarrow[\text{"Undo" Gauss-Jordan (reverse elimination)}]{} (\boxed{I_n} \mid A)$ so $BA = I_n$.

we get this matrix as the unique REF row equiv to B.