Lecture XI: §2.5 The Dot & Cross Products

§1. The dot product:

Def: Fix $\vec{u}, \vec{v}$ vectors. Then $\vec{u}\cdot\vec{v} = \underset{1\times n}{u^T}\ \underset{n\times 1}{v}$ (= a real number)

Call it the dot product of $\vec{u}$ & $\vec{v}$.

Eg: $\vec{u} = \begin{bmatrix}1\\-1\\1\end{bmatrix}$, $\vec{v} = \begin{bmatrix}0\\3\\5\end{bmatrix}$ $\quad \vec{u}\cdot\vec{v} = [1\ -1\ 1]\begin{bmatrix}0\\3\\5\end{bmatrix} = 0-3+5 = \boxed{2}$.

Algebraic properties: inherited from matrix operations

Fix $\vec{u}, \vec{v}, \vec{w}$ vectors, α in $\mathbb{R}$ scalar

(1) $\vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{u}$ [Commutative]

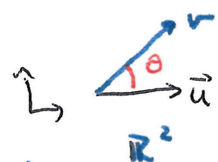
(2) $(\vec{u}+\vec{v})\cdot\vec{w} = \vec{u}\cdot\vec{w} + \vec{v}\cdot\vec{w}$ [Distributive]

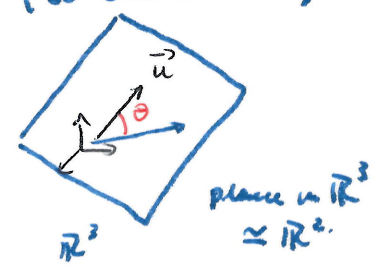
(3) $(\alpha\vec{u})\cdot\vec{v} = \vec{u}\cdot(\alpha\vec{v}) = \alpha(\vec{u}\cdot\vec{v})$ [Associative]

(4) $\vec{v}\cdot\vec{v} = \|\vec{v}\|^2$

Geometric form of the dot product:

- Fix $\vec{u}, \vec{v}$ & let θ be the angle between them (so $0 \leq \theta \leq \pi$)
- Set $a := \|\vec{u}\|$, $b := \|\vec{v}\|$



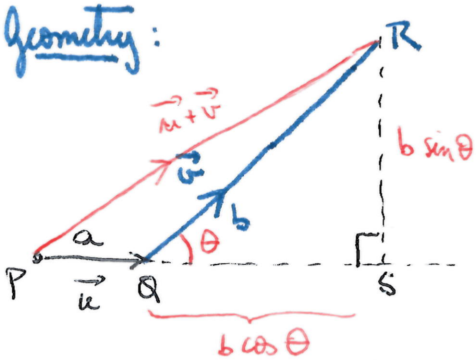


We compute $\|\vec{u}+\vec{v}\|^2$ in two different ways:

- **Algebra**: $\|\vec{u}+\vec{v}\|^2 = (\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v}) = \vec{u}\cdot\vec{u} + \vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{u} + \vec{v}\cdot\vec{v}$

$$= \|\vec{u}\|^2 + 2\,\vec{u}\cdot\vec{v} + \|\vec{v}\|^2 = \boxed{a^2+b^2+2\,\vec{u}\cdot\vec{v}} \quad \text{(I)}$$

- **Geometry**:



$$\|\vec{u}+\vec{v}\| = |PR|^2$$
$$= |PS|^2 + |SR|^2$$
$$= (a+b\cos\theta)^2 + (b\sin\theta)^2$$
$$= a^2 + 2ab\cos\theta + \underbrace{\boxed{b^2\cos^2\theta + b^2\sin^2\theta}}_{=b^2}$$
$$= \boxed{a^2+b^2+2ab\cos\theta} \quad \text{(II)}$$

Combine (I) & (II): $\vec{u}\cdot\vec{v} = ab\cos\theta$

Conclusion: $\boxed{\vec{u}\cdot\vec{v} = \|\vec{u}\|\,\|\vec{v}\|\cos\theta}$ works in $\mathbb{R}^2$ & $\mathbb{R}^3$

Use this to define angle between vectors for $\mathbb{R}^n$ with $n>3$.

Example: ① Pick $\vec{u}$ & $\vec{v}$ with lengths 3 & 6, respectively & angle $= 60°$ between them. Find $u\cdot\vec{v}$

Soln $\vec{u}\cdot\vec{v} = 3\cdot 6\cdot\cos 60° = \frac{18}{2} = \boxed{9}$

② Find the angle between $\begin{bmatrix}2\\1\\0\end{bmatrix}$ & $\begin{bmatrix}-1\\1\\5\end{bmatrix}$

Soln: $\cos\theta = \dfrac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|} = \dfrac{\begin{bmatrix}2 & 1 & 0\end{bmatrix}\begin{bmatrix}-1\\1\\5\end{bmatrix}}{\sqrt{5}\quad\sqrt{27}} = \dfrac{-1}{\sqrt{5}\,3\sqrt{3}} = \dfrac{-1}{3\sqrt{15}} \to \theta \approx 95$

Consequence Two vectors are perpendicular (or orthogonal) whenever the angle between them is $90°$. This is the <u>same</u> as having 0 as their dot product.

In symbols: $\vec{u}\perp\vec{v}$ if and only if $\vec{u}\cdot\vec{v}=0$.

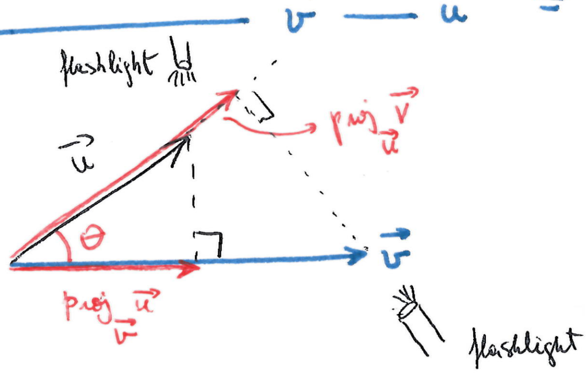
<u>In particular</u>: $\vec{0}\perp\vec{v}$ for any $\vec{v}$.

§2. Projections

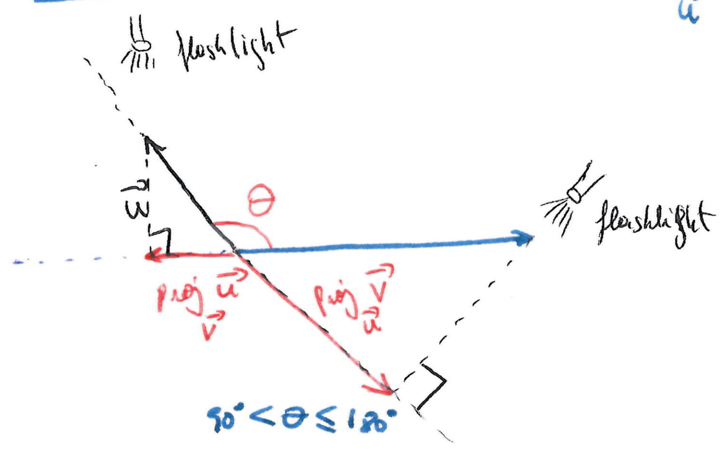
Fix $\vec{u},\vec{v}$ <u>nonzero</u> vectors. Two projections:

① orthogonal projection of $\vec{u}$ onto $\vec{v}$ = vector projection of $\vec{u}$ along $\vec{v} = \text{proj}_{\vec{v}}\vec{u}$

② ______ $\vec{v}$ — $\vec{u}$ = ______ $\vec{v}$ — $\vec{u} = \text{proj}_{\vec{u}}\vec{v}$



$0\leq\theta\leq 90°$



$90°<\theta\leq 180°$

$\text{proj}_{v}\vec{u}$ is a vector parallel to $\vec{v}$

- <u>signed</u> magnitude of $\text{proj}_{\vec{v}}\vec{u}$ (± magnitude) $=\begin{cases} |\text{proj}_{\vec{v}}\vec{u}| & \text{if } 0\leq\theta\leq 90° \text{ (same direction as } \vec{v}) \\ -|\text{proj}_{v}\vec{u}| & \text{if } 90°<\theta\leq 180° \text{ (opposite direction to } \vec{v}) \end{cases}$

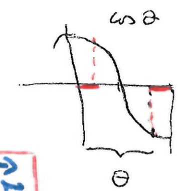
Relation: $$\text{proj}_{\vec v}\vec u = \underbrace{\boxed{\text{signed magnitude of } \text{proj}_{\vec v}\vec u}}_{\text{signed length}} \cdot \underbrace{\frac{\vec v}{\|\vec v\|}}_{\text{"direction"}}$$

(name $= \text{comp}_{\vec v}\vec u$)

Formula for $\text{comp}_{\vec v}\vec u = ?$ Use Trigonometry!

(1) $\text{comp}_{\vec v}\vec u = |\text{proj}_{\vec v}\vec u| = \|\vec u\| \cos\theta \qquad 0<\theta\leq 90°$

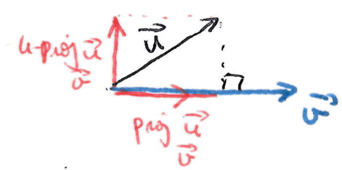
(2) $\text{comp}_{\vec v}\vec u = -|\text{proj}_{\vec v}\vec u| = -\|\vec u\| \cos(180°-\theta) = -\|\vec u\|(-\cos\theta) = \|\vec u\|\cos\theta$ if $0<\theta\leq 90°$



In both cases: $$\boxed{\text{comp}_{\vec v}\vec u} = \|\vec u\|\cos\theta = \|\vec u\| \frac{\vec u\cdot\vec v}{\|\vec u\|\,\|\vec v\|} = \boxed{\frac{\vec u\cdot\vec v}{\|\vec v\|}}$$

Conclusion: $$\boxed{\text{proj}_{\vec v}\vec u = \frac{\vec u\cdot\vec v}{\|\vec v\|^2}\,\vec v}$$

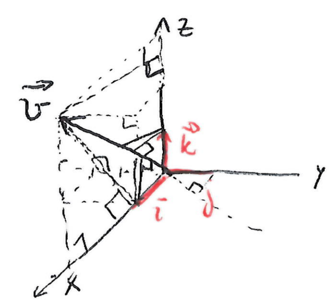
Application: $$\vec u = \underbrace{(\text{proj}_{\vec v}\vec u)}_{\text{parallel to } \vec v} + \underbrace{(\vec u - \text{proj}_{\vec v}\vec u)}_{\text{perpendicular to } \vec v}$$



It's the only way to decompose $\vec u$ as a sum $\vec w + \vec s$ with $\vec w \parallel \vec v$, $\vec s \perp \vec v$.

Example: Compute the projections of $\vec i, \vec j, \vec k$ along $\begin{bmatrix}1\\-1\\2\end{bmatrix}$.

$$\text{proj}_{\begin{bmatrix}1\\-1\\2\end{bmatrix}}\vec i = \frac{\begin{bmatrix}1&0&0\end{bmatrix}\begin{bmatrix}1\\-1\\2\end{bmatrix}}{\left\|\begin{bmatrix}1\\-1\\2\end{bmatrix}\right\|^2}\begin{bmatrix}1\\-1\\2\end{bmatrix} = \frac{1}{6}\begin{bmatrix}1\\-1\\2\end{bmatrix}$$



$$\text{proj}_{\begin{bmatrix}1\\-1\\2\end{bmatrix}}\vec j = \frac{\begin{bmatrix}0&1&0\end{bmatrix}\begin{bmatrix}1\\-1\\2\end{bmatrix}}{\left\|\begin{bmatrix}1\\-1\\2\end{bmatrix}\right\|^2}\begin{bmatrix}1\\-1\\2\end{bmatrix} = \frac{-1}{6}\begin{bmatrix}1\\-1\\2\end{bmatrix}$$

$$\text{proj}_{\begin{bmatrix}1\\-1\\2\end{bmatrix}}\vec j = \frac{\begin{bmatrix}0&0&1\end{bmatrix}\begin{bmatrix}1\\-1\\2\end{bmatrix}}{\left\|\begin{bmatrix}1\\-1\\2\end{bmatrix}\right\|^2}\begin{bmatrix}1\\-1\\2\end{bmatrix} = \frac{2}{6}\begin{bmatrix}1\\-1\\2\end{bmatrix} = \frac{1}{3}\begin{bmatrix}1\\-1\\2\end{bmatrix}$$

$$\text{comp}_{\begin{bmatrix}1\\-1\\2\end{bmatrix}}\vec i = \frac{1}{\sqrt6}, \quad \text{comp}_{\begin{bmatrix}1\\-1\\2\end{bmatrix}}\vec j = \frac{-1}{\sqrt6}, \quad \text{comp}_{\begin{bmatrix}1\\-1\\2\end{bmatrix}}\vec k = \frac{2}{\sqrt6}.$$

§3 Cross Product: ONLY for $\mathbb{R}^3$.

Def: Fix $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ two vectors in $\mathbb{R}^3$.

The <u>cross product</u> of $\vec{u}$ & $\vec{v}$ is a vector in $\mathbb{R}^3$ defined as

$$\vec{u} \times \vec{v} = \begin{bmatrix} u_2 v_3 - u_3 v_2 \\ -(u_1 v_3 - u_3 v_1) \\ u_1 v_2 - u_2 v_1 \end{bmatrix}$$

Determinantal formula = ?

<u>Recall</u>: determinant of a 2x2 matrix $\det \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - b_1 a_2$

- Determinant of a 3x3 matrix:

$$\det \begin{vmatrix} x & y & z \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = x \det \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - y \det \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + z \det \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$$

Using these definitions, we get:

remember the sign change!

$$\vec{u} \times \vec{v} = \begin{array}{c} \text{Basic unit vect.} \\ \vec{u} \\ \vec{v} \end{array} \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \underbrace{\det \begin{pmatrix} u_2 & u_3 \\ v_2 & v_3 \end{pmatrix}}_{\text{scalar}} \vec{i} - \underbrace{\det \begin{pmatrix} u_1 & u_3 \\ v_1 & v_3 \end{pmatrix}}_{\text{scalar}} \vec{j} + \underbrace{\det \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}}_{\text{scalar}} \vec{k}.$$

<u>Example</u>: $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 3 \\ 2 & -1 & 2 \end{vmatrix} = \vec{i}\,(2^2 - (-1)3) - (1\cdot 2 - 2\cdot 3)\,\vec{j} + (-1 - 2\cdot 2)\,\vec{k}$$
$$= 7\cdot\vec{i} - (-4)\,\vec{j} + (-5)\,\vec{k} = \begin{bmatrix} 7 \\ 4 \\ -5 \end{bmatrix}.$$

<u>Properties</u>: $\vec{u}, \vec{v}, \vec{w}$ in $\mathbb{R}^3$, α, β scalars

(1) $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$ [<u>ANTI</u>COMMUTATIVE], so $\vec{u} \times \vec{u} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ for all $\vec{u}$.

(2) $\alpha\vec{u} \times \beta\vec{v} = (\alpha\beta)(\vec{u} \times \vec{v})$ [ASSOCIATIVE], so $\vec{0} \times \vec{u} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ for all $\vec{u}$

(3) $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$ [DISTRIBUTIVE]

(4) $(\vec{u} + \vec{v}) \times \vec{w} = \vec{u} \times \vec{w} + \vec{v} \times \vec{w}$.

(5) $\vec{u} \cdot (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}) \cdot \vec{w}$.

<u>Proof</u>: (1) & (2) Follows from property of 2x2 det $\det \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc = -\det \begin{vmatrix} c & d \\ a & b \end{vmatrix}$

$\det \begin{vmatrix} \alpha a & \alpha b \\ c & d \end{vmatrix} = \alpha \det \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

(4) = Combine (1) & (3).

(3) Also follows from prop of 2×2-det: $\det\begin{vmatrix} a+a' & b+b' \\ c & d\end{vmatrix} = (a+a')d - c(b+b')$

$= (ad-cb)+(a'd-cb')$

$= \det\begin{vmatrix} a & b \\ c & d\end{vmatrix} + \det\begin{vmatrix} a' & b' \\ c & d\end{vmatrix}.$

(5) direct computation using the formulas.

Key Proposition: $\vec{u}\times\vec{v} \perp \vec{u}$ & $\vec{u}\times\vec{v}\perp\vec{v}$

Proof: We must verify $\vec{u}\cdot(\vec{u}\times\vec{v}) = \vec{v}\cdot(\vec{u}\times\vec{v}) = 0.$

By (5), $\bullet\ \vec{u}\cdot(\vec{u}\times\vec{v}) = \underbrace{(\vec{u}\times\vec{u})}_{=\vec{0}\text{ by }(1)}\cdot\vec{v} = \vec{0}\cdot\vec{v} = 0$ ✓

$\bullet\ \vec{v}\cdot(u\times\vec{v}) = -\vec{v}\cdot(\vec{v}\times\vec{u}) = -\underbrace{(\vec{v}\times\vec{v})}_{=\vec{0}}\cdot\vec{u} = -\vec{0}\cdot\vec{u} = 0$ ✓