

§1 Cross product in \mathbb{R}^3 :

\vec{u}, \vec{v} in $\mathbb{R}^3 \implies \vec{u} \times \vec{v}$ is a vector in $\mathbb{R}^3 = \det \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{bmatrix} u_2 v_3 - u_3 v_2 \\ -(u_1 v_3 - u_3 v_1) \\ u_1 v_2 - u_2 v_1 \end{bmatrix}$

Example: $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \vec{v} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}, \vec{u} \times \vec{v} = \det \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ 2 & -1 & 2 \end{vmatrix} = \hat{i}(2^2 - (-1)3) - (1 \cdot 2 - 2 \cdot 3)\hat{j} + (-1 - 2^2)\hat{k}$
 $= 7\hat{i} - (-4)\hat{j} + (-5)\hat{k} = \begin{bmatrix} 7 \\ 4 \\ -5 \end{bmatrix}$

Properties: $\vec{u}, \vec{v}, \vec{w}$ in $\mathbb{R}^3, \alpha, \beta$ scalars

(1) $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$ [ANTICOMMUTATIVE], so $\vec{u} \times \vec{u} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ for all \vec{u} .

(2) $\alpha \vec{u} \times \beta \vec{v} = (\alpha\beta)(\vec{u} \times \vec{v})$ [ASSOCIATIVE], so $\vec{0} \times \vec{u} = \vec{0}$ for all \vec{u} .

(3) $\left. \begin{aligned} \vec{u} \times (\vec{v} + \vec{w}) &= \vec{u} \times \vec{v} + \vec{u} \times \vec{w} \\ (\vec{u} + \vec{v}) \times \vec{w} &= \vec{u} \times \vec{w} + \vec{v} \times \vec{w} \end{aligned} \right\}$ [DISTRIBUTIVE]

(4) $\vec{u} \cdot (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}) \cdot \vec{w}$

Proof: (1), (2) & (3) follow from properties of 2x2 determinants:

• $\det \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc = -\det \begin{vmatrix} c & d \\ a & b \end{vmatrix}$

• $\det \begin{vmatrix} \alpha a & \alpha b \\ \beta c & \beta d \end{vmatrix} = (\alpha a)(\beta d) - (\alpha b)(\beta c) = \alpha\beta(ad - bc) = \alpha\beta \det \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

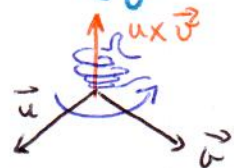
• $\det \begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = (a+a')d - (b+b')c = (ad - bc) + (a'd - b'c) = \det \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \det \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$

(4) is a direct computation.

• Key Property: $\vec{u} \times \vec{v} \perp \vec{u}$ & $\vec{u} \times \vec{v} \perp \vec{v}$.

Why? Use dot product
 $\vec{u} \cdot (\vec{u} \times \vec{v}) = (\vec{u} \times \vec{u}) \cdot \vec{v} = \vec{0} \cdot \vec{v} = 0$
 $\vec{v} \cdot (\vec{u} \times \vec{v}) = (+1) \vec{v} \cdot (\vec{u} \times \vec{u}) = -(\vec{v} \times \vec{v}) \cdot \vec{u} = \vec{0} \cdot \vec{u} = 0.$

• Direction of $\vec{u} \times \vec{v}$ is given by the right-hand rule
 (or has no direction is $\vec{u} \times \vec{v} = \vec{0}$)



• Geometry of cross products: If $\vec{u}, \vec{v} \neq \vec{0}$, fix $\theta =$ angle between \vec{u} & \vec{v} .

Then $\|\vec{u} \times \vec{v}\| = \|\vec{u}\| \|\vec{v}\| \sin \theta$

Proof: Write $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$

Verify that $\|\vec{u} \times \vec{v}\|^2 + (\vec{u} \cdot \vec{v})^2 = \|\vec{u}\|^2 \|\vec{v}\|^2$ (simple computation)

Then, since $\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos \theta$, we set

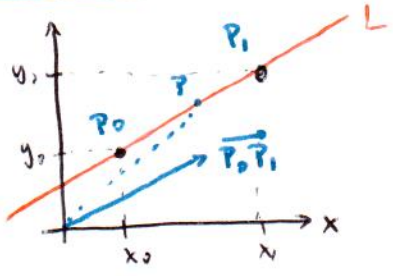
$$\|\vec{u} \times \vec{v}\|^2 + \|\vec{u}\|^2 \|\vec{v}\|^2 \cos^2 \theta = \|\vec{u}\|^2 \|\vec{v}\|^2$$

$$\|\vec{u} \times \vec{v}\|^2 = \|\vec{u}\|^2 \|\vec{v}\|^2 (1 - \cos^2 \theta) = \|\vec{u}\|^2 \|\vec{v}\|^2 \sin^2 \theta \quad (*)$$

since $0 \leq \sin \theta$ for $0 \leq \theta \leq 180^\circ$, we have $\|\vec{u} \times \vec{v}\| = \|\vec{u}\| \|\vec{v}\| \sin \theta$
by taking $\sqrt{\quad}$ on both sides of (*)

Conclusion: (1) We know the magnitude & direction of $\vec{u} \times \vec{v}$
(2) $\vec{u} \times \vec{v} = \vec{0}$ for $\vec{u}, \vec{v} \neq \vec{0}$ if and only if $\theta = 0$ or 180° ($\vec{u} \parallel \vec{v}$)

§2. Lines in \mathbb{R}^2 & \mathbb{R}^3 :



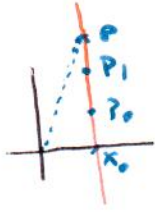
$P_0 = (x_0, y_0)$
 $P_1 = (x_1, y_1)$

• Assume $x_0 \neq x_1$
The equation of the line L through P_0 & P_1 is

$$y = m(x - x_0) + y_0$$

with $m = \text{slope} = \frac{y_1 - y_0}{x_1 - x_0}$

• If $x_0 = x_1$, the line is vertical & has equation: $x = x_0$



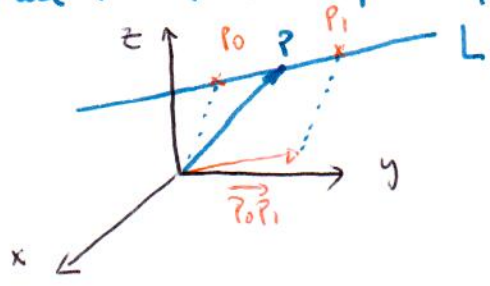
Parametric or vector form : ?

• direction of L : $\vec{P_0P_1}$

$$\vec{OP} = t \vec{P_0P_1} + \vec{OP_0} \quad \text{for some } t \text{ in } \mathbb{R}$$

(works for any slope!)

Can use this idea to find equations describing lines in \mathbb{R}^3 .



• Two points (P_0, P_1) in \mathbb{R}^3 uniquely determine a line

• Vector Eqn: $\vec{OP} = t \vec{P_0P_1} + \vec{OP_0}$ for $t \in \mathbb{R}$

• P lies on L if and only if $\vec{PP_0}$ is parallel to $\vec{P_0P_1}$
 $\vec{P_0P_1} = \vec{OP} - \vec{OP_0}$ (if parallel, then $= t \vec{P_0P_1}$)

Parametric equation = ? Use the components of the vectors

Write $P_0 = (x_0, y_0, z_0)$ & \vec{u} = direction of $L = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

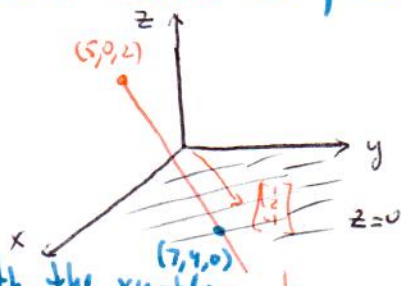
If $P = (x, y, z)$, then P lies in L if and only if $\vec{P_0P} = t\vec{u}$
for some t .

In coordinates: $\begin{bmatrix} x-x_0 \\ y-y_0 \\ z-z_0 \end{bmatrix} = t \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

so $\begin{cases} x-x_0 = ta \\ y-y_0 = tb \\ z-z_0 = tc \end{cases}$ for some $t \iff \begin{cases} x = x_0 + at \\ y = y_0 + bt \\ z = z_0 + ct \end{cases}$ for some t

Example: Find the equation of the line which is parallel to $\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$ & passes through $(5, 0, 2)$

A: $\begin{cases} x = 5 + t \\ y = 0 + t \\ z = 2 - t \end{cases}$



Find the intersection of this line with the xy -plane.

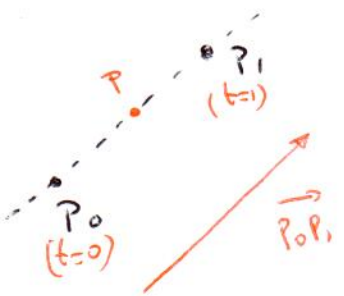
xy -plane: $z=0$

$\implies 2-t=0$ so $t=2$

Substitute t in other 2 components

$\left. \begin{matrix} x = 5 + 2 = 7 \\ y = 0 + 2 = 2 \end{matrix} \right\} \underline{A}: Pt = (7, 2, 0)$

§3 Line Segments:



We restrict the equation of the line through P_0 & P_1 to specific values of t

$\vec{P_0P} = t\vec{P_0P_1}$ for $0 \leq t \leq 1$

Why this range? $\cdot P = P_0$ for $t=0$ ($0 = \vec{P_0P_0} = 0 \cdot \vec{P_0P_1}$)
 $\cdot P = P_1$ for $t=1$ ($\vec{P_0P_1} = 1 \cdot \vec{P_0P_1}$)

So points in between correspond to values of t in between 0 & 1 .

Special case: $t = \frac{1}{2} \implies$ midpoint between P_0 & P_1 (last time)
 $P_0 = \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix}, P_1 = \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix}$ midpoint = $\begin{bmatrix} \frac{a_1+b_1}{2} \\ \frac{b_1+b_2}{2} \\ \frac{c_1+c_2}{2} \end{bmatrix} \implies \|\vec{P_0P}\| = \frac{1}{2} \|\vec{P_0P_1}\|$