Lecture XIII: §2.4 (cont) Lines & planes.

Recall: Lines L in $\mathbb{R}^n$ have vector form equation:

$$P \text{ in } L \equiv \boxed{\overrightarrow{OP} = t\,\vec{v} + \overrightarrow{OP_0} \quad \text{for } t \text{ in } \mathbb{R}} \qquad \vec{v} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \quad \overrightarrow{OP_0} = \begin{bmatrix} p_1 \\ \vdots \\ p_n \end{bmatrix} \quad \overrightarrow{OP} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

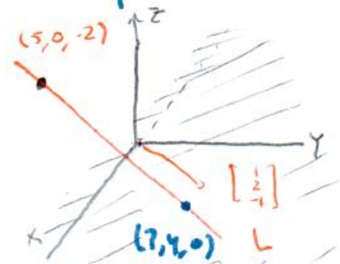
For $n=3$ we write down parametric equations (components of vector equation)

$$\begin{cases} x = t\,\boxed{a} + \boxed{x_0} \\ y = t\,\boxed{b} + \boxed{y_0} \\ z = t\,\boxed{c} + \boxed{z_0} \end{cases}$$

direction of L pt P_0

Example: • Find the equation of the line which is parallel to $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ & passes through $(5, 0, +2)$

$$\begin{cases} x = 5 + t \\ y = 0 + 2t \\ z = +2 + (-t) \end{cases}$$



• Find the intersection of this line with the xy-plane:

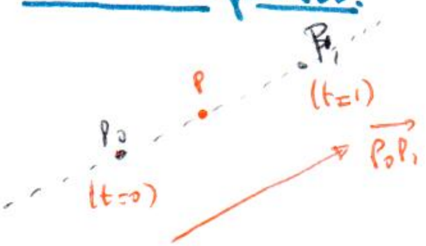
xy-plane : $\boxed{z = 0}$

⇝ Substitute param equation in plane eqn to solve for t.

$2 - t = 0$ so $\boxed{t = 2}$

• Substitute value of t in x, y components: $\left.\begin{matrix} x = 5 + 2 = 7 \\ y = 0 + 2 \cdot 2 = 4 \end{matrix}\right\}$ $A = \text{pt} = (7, 4, 0)$

§ Line Segments:



We restrict the vector equation of the line through P_0 & P_1 to specific values of t

$$\boxed{\overrightarrow{OP} = t\,\overrightarrow{P_0P_1} + \overrightarrow{OP_0} \quad \text{for } 0 \leq t \leq 1}$$

Why this range?

Equiv $\underbrace{\overrightarrow{OP} - \overrightarrow{OP_0}}_{= \overrightarrow{P_0P}} = t\,\overrightarrow{P_0P_1}$.

$\left.\begin{matrix} \text{• For } P = P_0 \text{ we use } t = 0 \\ \text{• For } P = P_1 \text{ ——— } t = 1 \end{matrix}\right\}$ so points in between correspond to values of t between 0 & 1

Special case: $t = \frac{1}{2}$ midpoint between P_0 & P_1

$P_0 = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$, $P_1 = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$ ⇝ midpoint $= P = \left(\frac{a_1+b_1}{2}, \dots, \frac{a_n+b_n}{2}\right)$

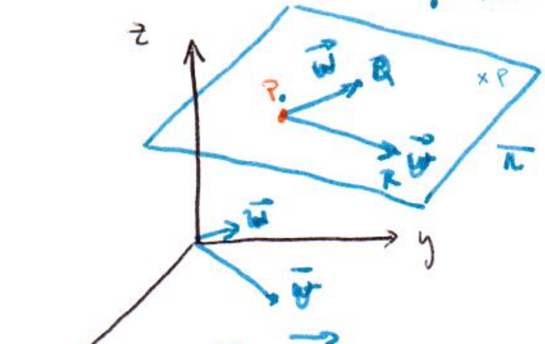
$\boxed{\overrightarrow{P_0P} = \frac{1}{2}\overrightarrow{P_0P_1}}$

§2. Planes in 3-Spaces

Two ways to determine a plane in 3-space

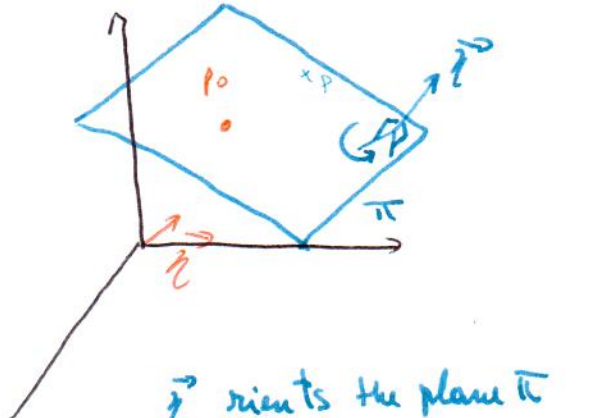
① A point P_0 & 2 non-parallel directions $(\vec{v}, \vec{w})$

[Equivalently: 3 non-collinear pts P_0, Q, R]



$\vec{w} = \overrightarrow{P_0Q}$
$\vec{v} = \overrightarrow{P_0R}$

② A point P_0 & a normal $\vec{n}$



- $\vec{n}$ orients the plane π
- $\vec{n}$ is normal to π if $\vec{n}$ is perpendicular to the 2 directions of π

$$\boxed{\vec{n} = \vec{v} \times \vec{w}}$$

($\underbrace{\vec{w} \times \vec{v}}_{= -\vec{n}}$ also works)

We know $\vec{n} \cdot \vec{v} = \vec{n} \cdot \vec{w} = 0$

- Vector equation for π: $\overrightarrow{P_0P} \cdot \vec{n} = 0$

Explicitly: $\left.\begin{array}{l} P = (x, y, z) \\ P_0 = (x_0, y_0, z_0) \\ \vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \neq \vec{0} \end{array}\right\}$ $\boxed{a(x-x_0) + b(y-y_0) + c(z-z_0) = 0}$

$$ax + by + cz = \underbrace{ax_0 + by_0 + cz_0}_{\text{fixed } \#.}$$

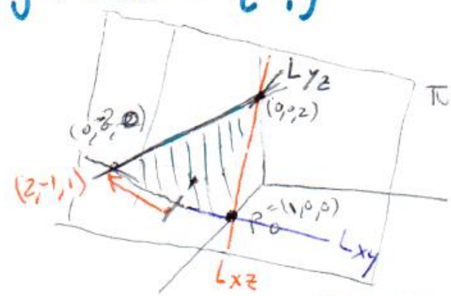
- Conversely, from the coefficients of the equation we get $\vec{n}$
  - any explicit solution gives P_0.

Example: Find the equation of the plane passing through $P_0 = (1, 0, 0)$, $R_0 = (1, 1, 1)$, $Q_0 = (2, 1, -1)$.
Compute the intersection of this plane with the 3 coordinate planes (xy-, xz- & yz-planes)

2 directions ; $\vec{v} = \overrightarrow{P_0Q_0} = \begin{bmatrix}2\\1\\-1\end{bmatrix} - \begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}1\\1\\-1\end{bmatrix}$

$\vec{w} = \overrightarrow{P_0R_0} = \begin{bmatrix}1\\1\\1\end{bmatrix} - \begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}0\\1\\1\end{bmatrix}$

So $\vec{\eta} = \vec{v}\times\vec{w} = \det\begin{vmatrix} i & j & k\\ 1 & 1 & -1\\ 0 & 1 & 1\end{vmatrix} = \vec{i}\,(1-(-1)) - \vec{j}\cdot 1 + \vec{k} = \begin{bmatrix}2\\-1\\1\end{bmatrix}$



Equation : $2(x-\underset{P_0}{1}) + (-1)(y-0) + (z-0) = 0$

$$\boxed{2x - y + z = 2}$$

- Easy check : P_0, Q_0 & R_0 satisfy the equation
- The 3 intersections will be lines in the corresponding coordinate planes.

XY-plane : $\begin{cases} z = 0\\ 2x - y + z = 2\end{cases}$ line $L_{xy} = 2x - y = 2$ in xy-plane

YZ-plane : $\begin{cases} x = 0\\ 2x - y + z = 2\end{cases}$ — $L_{yz} = z - y = 2$ — yz-plane

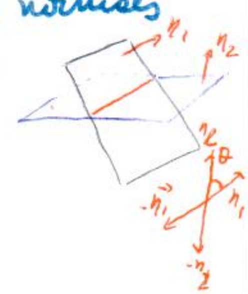
XZ-plane $\begin{cases} y = 0\\ 2x - y + z = 2\end{cases}$ — $L_{xz} = 2x + z = 2$ — xz-plane

3 intersections between the lines $(0,0,2)$, $(0,-2,0)$ & $(1,0,0)$

§3 Parallel and orthogonal planes:

Def : Angles between 2 planes = (acute) angle between their normals

(Use $|\vec{\eta}_1 \cdot \vec{\eta}_2| = \|\vec{\eta}_1\|\,\|\vec{\eta}_2\|\,|\cos\theta|$ to find θ with $0 \le \theta \le 90^\circ$



In particular : $(\theta = 0)$ parallel planes : $\vec{n}_1 \,/\!/\, \vec{n}_2$

$(\theta = 90^\circ)$ orthogonal planes : $\vec{n}_1 \perp \vec{n}_2$.

Example : Find the parallel plane to $3x - 2y + 5z = 4$ passing through $(1,-1,1)$

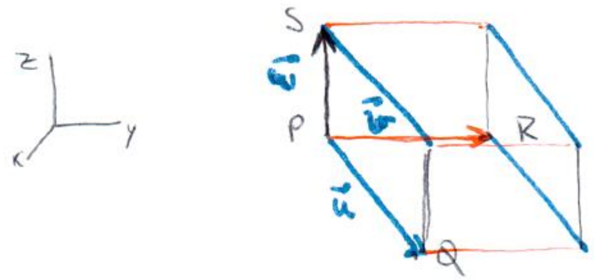
Soln $\vec{\eta} = \begin{bmatrix}3\\-2\\5\end{bmatrix}$ π : $3(x-1) + (-2)(y+1) + 5(z-1) = 0$

$$\boxed{3x - 2y + 5z = 10}$$

Example : Find a plane orthogonal to $3x - 2y + 5z = 4$, passing through $P_0 = (1,0,0)$ & $Q_0 = (1,1,0)$

Soln : normal $\vec{\eta} \perp \begin{bmatrix}3\\-2\\5\end{bmatrix}$; — $\vec{\eta} \perp \overrightarrow{P_0Q_0} = \begin{bmatrix}0\\1\\0\end{bmatrix}$ $\rightsquigarrow$ $\vec{\eta} = \det\begin{vmatrix} i & j & k\\ 3 & -2 & 5\\ 0 & 1 & 0\end{vmatrix} = \begin{bmatrix}-5\\0\\3\end{bmatrix}$ π : $\boxed{-5x + 3z = -5}$

§4: Coplanar points: 4 points P, Q, R, S are coplanar if and only if the 3 vectors $\overrightarrow{PQ}$, $\overrightarrow{PR}$ & $\overrightarrow{PS}$ form a flat parallelepiped. (ie Volume = 0)



$$\text{Vol} = \| \vec{u} \cdot (\vec{v} \times \vec{w}) \| = \| (\vec{u} \times \vec{v}) \cdot \vec{w} \|$$

$\vec{u} = \overrightarrow{PS}$
$\vec{v} = \overrightarrow{PR}$
$\vec{w} = \overrightarrow{PS}$

Example: Show that $\underset{P}{(1,3,2)}$, $\underset{Q}{(3,-1,6)}$, $\underset{R}{(5,2,0)}$, $\underset{S}{(3,6,-4)}$ are coplanar

Soln 1: $\vec{u} = \begin{bmatrix} 2 \\ -4 \\ 4 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 4 \\ -1 \\ -2 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} 2 \\ 3 \\ -6 \end{bmatrix}$ $\rightsquigarrow \text{Vol} = \begin{bmatrix} 2 \\ -4 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 12 \\ 20 \\ 14 \end{bmatrix} = 0$ ✓

$$\vec{v} \times \vec{w} = \begin{vmatrix} i & j & k \\ 4 & -1 & -2 \\ 2 & 3 & -6 \end{vmatrix} = \begin{bmatrix} 12 \\ 20 \\ 14 \end{bmatrix}$$

Soln 2: Find the equation of the plane through a choice of 3 pts & check 4th point also verifies the equation.
(P, R, S)

$\vec{n} = \vec{v} \times \vec{w} = \begin{bmatrix} 12 \\ 20 \\ 14 \end{bmatrix}$ $\quad \pi : \; 12x + 20y + 14z = 12 \cdot 1 + 20 \cdot 3 + 14 \cdot 2 = 100$

Check for Q: $\quad 12 \cdot 3 - 20 \cdot 1 + 14 \cdot 6 \stackrel{?}{=} 100$ ✓