Lecture XIV §3.1-2: Intro & Vector Space properties of $\mathbb{R}^n$
§3.3: Examples of subspaces of $\mathbb{R}^n$

§1. The vector space $\mathbb{R}^n$ & its properties:

- Vectors in $\mathbb{R}^2, \mathbb{R}^3, \dots, \mathbb{R}^n$
- Solutions to homogeneous systems in $\mathbb{R}^n$

$$\underline{x} = x_{i_1}\begin{bmatrix} \\ \\ \end{bmatrix} + x_{i_2}\begin{bmatrix} \\ \\ \end{bmatrix} + \dots + x_{i_j}\begin{bmatrix} \\ \\ \end{bmatrix}$$
(lin comb. of vectors in $\mathbb{R}^n$)

Q: What properties do they have?

① $\vec{0}$ lies in the space

② add vectors / solutions & we remain a vector / solution

③ scalar multiplication preserves $\mathbb{R}^n$ / space of solutions

Note: . ①-③ fail for non-homogeneous systems.
. These properties will characterize vector spaces ($\mathbb{R}^n$ will be our favorite example)

Theorem 1: Write $\mathbb{V} = \mathbb{R}^n$. For x, y, z in $\mathbb{V}$, α, β scalars, we have the following properties:

- Closure Properties: (C1) x, y in $\mathbb{V}$, then $x+y$ in $\mathbb{V}$
  (②, ③ above) (C2) x in $\mathbb{V}$, αx in $\mathbb{V}$

- Addition properties:
  (A1) $x+y = y+x$ [COMMUTATIVE]
  (A2) $x+(y+z) = (x+y)+z$ [ASSOCIATIVE]
  (A3) $\vec{\mathbb{O}}$ in $\mathbb{V}$ satisfies $x + \mathbb{O} = \mathbb{O} + x = x$ [NEUTRAL ELEMENT]
  (A4) Given x in $\mathbb{V}$, we can find y in $\mathbb{V}$ with $x + y = \mathbb{O}$ ($y = $ "$-x$" $= (-1)\cdot x$) [ADDITIVE INVERSE]

- Scalar Multiplication properties:
  (M1) $\alpha(\beta x) = (\alpha\beta)x$ [ASSOCIATIVE]
  (M2) $\alpha(x+y) = \alpha x + \alpha y$ [DISTRIBUTIVE I]
  (M3) $(\alpha+\beta)x = \alpha x + \beta x$ [———— II]
  (M4) $1 \cdot x = x$ for all x in $\mathbb{V}$.

Note: (A4) follows from C2

Def: A subset $\mathbb{W}$ of $\mathbb{R}^n$ satisfying these 10 properties is called a (vector) subspace of $\mathbb{R}^n$.

Note: . A1, A2, M1 — M4 are inherited from $\mathbb{R}^n$, so we only need to check

$\boxed{C_1, C_2, A3}$ & A4. But A4 will follow from C2 because $-X = (-1)X$.

Theorem 2: A subset $\mathbb{V}$ of $\mathbb{R}^n$ is a subspace if and only if:

(S1) The zero vector $\vec{0}$ is in $\mathbb{V}$

(S2) $x+y$ lies in $\mathbb{V}$ whenever x, y are in $\mathbb{V}$

(S3) αx ———— x is in $\mathbb{V}$ & α is any scalar in $\mathbb{R}$.

§2. Examples / Non-examples:

Meta example: A solution set to a homogeneous system of m equations in n unknowns.

. Write it as $\{\underline{x} \text{ in } \mathbb{R}^n \mid A \cdot \underline{x} = \underline{0}\}$

(S1) The Trivial soln is $\underline{0}$ ✓

(S2) $A(\underline{x}+\underline{y}) = A\underline{x} + A\underline{y} = \underline{0} + \underline{0} = \underline{0}$ ✓ $\rightsquigarrow$ $\underline{x}+\underline{y}$ is a soln.

(S3) $A(\alpha \underline{x}) = \alpha(A\underline{x}) = \alpha \cdot \underline{0} = \underline{0}$ ✓ $\rightsquigarrow$ $\alpha \underline{x}$ ————.

Example 1. A line L in $\mathbb{R}^3$ through $(0,0,0)$, e.g. $\begin{cases} x+y+z=0 \\ x+2y-z=0 \end{cases}$

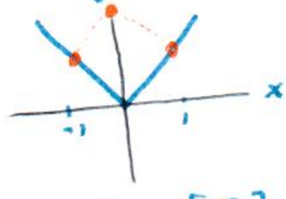
(S1) $[0,0,0]^t$ in the line ✓

(S2) $[x_1, y_1, z_1]^t$ in L, $[x_2, y_2, z_2]^t$ ———— $\rightsquigarrow$ $\begin{cases} .\ (x_1+x_2)+(y_1+y_2)+(z_1+z_2) = (x_1+y_1+z_1) + (x_2+y_2+z_2) = 0+0=0 \\ .\ (x_1+x_2)+2(y_1+y_2)-(z_1+z_2) = (x_1+2y_1-z_1) + (x_2+2y_2-z_2) = 0+0=0 \end{cases}$

So $[x_1+x_2, y_1+y_2, z_1+z_2]^t$ in L

(S3) $\left.\begin{matrix} \alpha \text{ in } \mathbb{R} \\ [x_1, y_1, z_1]^t \text{ in } L \end{matrix}\right\}$ $\rightsquigarrow$ $\begin{cases} .\ \alpha x_1 + \alpha y_1 + \alpha z_1 = \alpha(x_1+y_1+z_1) = \alpha \cdot 0 = 0 \\ .\ \alpha x_1 + 2(\alpha y_1) - \alpha z_1 = \alpha(x_1+2y_1-z_1) = \alpha \cdot 0 = 0 \end{cases}$

So $\alpha [x_1, y_1, z_1]^t$ in L.



Non-example: graph of $|x|$ in $\mathbb{R}$

(S1) holds

(S2) fails $\underline{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\underline{y} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ $\rightsquigarrow$ $\underline{x}+\underline{y} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$ not in the graph!

Non-example 2: $\mathbb{W} = \{\underline{x} : \underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \ x_1, x_2 \text{ in } \mathbb{R}\}$. = plane with equation $x_3 = 3$

(S1) fails

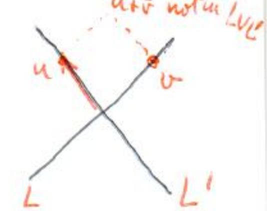
Non-example 3: $\mathbb{W} = \{ \underline{x} : x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad x_1, x_2 \text{ integers} \}$

(S1) holds

(S2) —— (sum of integers is an integer)

(S3) does NOT hold $\quad \frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$ not in $\mathbb{W}$, but $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is.

Non-example 4: Union of 2 different lines in $\mathbb{R}^2$ through $(0,0)$



§2. The span of a subset:

Def: given vectors $\vec{v}_1, \dots, \vec{v}_r$ in $\mathbb{R}^n$, we write

$$\mathbb{W} = \mathrm{Sp}(\vec{v}_1, \dots, \vec{v}_r) = \text{set of } \underline{\text{all}} \text{ linear combinations } \vec{v}_1, \dots, \vec{v}_r$$

$$= \{ \alpha_1 \vec{v}_1 + \dots + \alpha_r \vec{v}_r \ : \ \alpha_1, \dots, \alpha_r \text{ in } \mathbb{R} \}$$

Theorem 2: The set $\mathbb{W} = \mathrm{Sp}(\vec{v}_1, \dots, \vec{v}_r)$ is a subspace of $\mathbb{R}^n$

Proof: (S1) $\vec{0} = 0 \cdot \vec{v}_1 + \dots + 0 \cdot \vec{v}_r$ ✓

(S2) $\vec{x}$ in $\mathbb{W}$: $\vec{x} = a_1 \vec{v}_1 + \dots + a_r \vec{v}_r \qquad a_1, \dots, a_r$ scalars

$\vec{y}$ " " $\qquad \vec{y} = b_1 \vec{v}_1 + \dots + b_r \vec{v}_r \qquad b_1, \dots, b_r$ ——

$\vec{x} + \vec{y} = (a_1 + b_1)\vec{v}_1 + \dots + (a_r + b_r)\vec{v}_r \qquad (a_1+b_1), \dots, (a_r+b_r)$ scalars.

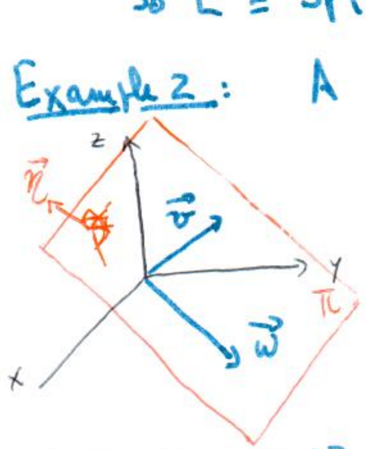
(S3) $\alpha \vec{x} = (\alpha a_1)\vec{v}_1 + \dots + (\alpha a_r)\vec{v}_r \qquad \alpha a_1, \dots, \alpha a_r$ scalars.

Example 1: A line L through the origin in $\mathbb{R}^n$: $\qquad v$ = direction of L

Vector equation $\overrightarrow{OP} = \vec{0} + t\vec{v} = t\vec{v}$

So $L = \mathrm{Sp}(\vec{v})$.

Example 2: A plane π in $\mathbb{R}^3$ passing through the origin.



Eqn $\overrightarrow{OP} \cdot \vec{n} = 0$ $\qquad \vec{n} = \vec{v} \times \vec{w}$

$\vec{v}, \vec{w}$ non-collinear vectors $\qquad \overrightarrow{OP} \cdot \vec{n} = 0$ becomes

$\overrightarrow{OP} = \alpha \vec{v} + \beta \vec{w}$

Conclude: $\pi = \mathrm{Sp}(\vec{v}, \vec{w})$

Why? $\vec{v} = \begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix}, \vec{w} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \rightsquigarrow \vec{n} = \vec{v} \times \vec{w} = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} \qquad$ Eqn: $-x + 2y - z = 0$

- $\vec{y}$ in $Sp(\vec{v}, \vec{w})$ so $y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = a\begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2a+b \\ -a+b \\ b \end{bmatrix}$
- Given $\vec{y}$ we must find a, b that solves the system of 3 equations in 2 unknowns

Augmented matrix $\left[\begin{array}{cc|c} -2 & 1 & y_1 \\ -1 & 1 & y_2 \\ 0 & 1 & y_3 \end{array}\right] \xrightarrow[R_1 \to -R_1]{R_2 \leftrightarrow R_1} \left[\begin{array}{cc|c} 1 & -1 & -y_2 \\ -2 & 1 & y_1 \\ 0 & 1 & y_3 \end{array}\right] \xrightarrow{R_2 \to R_2 + 2R_1} \left[\begin{array}{cc|c} 1 & -1 & -y_2 \\ 0 & -1 & y_1 - 2y_2 \\ 0 & 1 & y_3 \end{array}\right]$

$\xrightarrow[R_2 \to -R_2]{R_3 \to R_3 + R_2} \left[\begin{array}{cc|c} 1 & -1 & -y_2 \\ 0 & 1 & -y_1 + 2y_2 \\ 0 & 0 & \boxed{y_1 - 2y_2 + y_3} \end{array}\right]$

The system is consistent if and only if $y_1 - 2y_2 + y_3 = 0$ $(= -\text{Eqn of } \pi)$.

§4. The Null Space of a Matrix:

Def: Given an $m \times n$ matrix A, the solution set to $A \cdot \underline{x} = \underline{0}$ in $\mathbb{R}^n$ is the Null Space (or Kernel) of A:

$$\mathcal{N}(A) = \{x : Ax = 0, \ x \text{ in } \mathbb{R}^n\}$$

Theorem 3: $\mathcal{N}(A)$ is a subspace of $\mathbb{R}^n$.

Observe: $\mathcal{N}(A) = \{ x_{i_1} \underbrace{\begin{bmatrix} \\ \end{bmatrix}}_{\vec{v}_1} + \cdots + x_{i_r} \underbrace{\begin{bmatrix} \\ \end{bmatrix}}_{\vec{v}_r} \} = Sp(\vec{v}_1, \ldots, \vec{v}_r)$

Example $A = \begin{bmatrix} 1 & 1 & 3 & 1 \\ 3 & 2 & 8 & 5 \\ -1 & -2 & -4 & 1 \end{bmatrix} \rightsquigarrow [A|0] \underset{\text{row equiv}}{\sim} \left[\begin{array}{cccc|c} & & \text{indep.} & \text{indep.} & \\ 1 & 0 & 2 & 3 & 0 \\ 0 & 1 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

$\begin{cases} x_1 = -2x_3 - 3x_4 \\ x_2 = x_3 + 2x_4 \end{cases} \rightsquigarrow \vec{x} = x_3 \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ 2 \\ 0 \\ 1 \end{bmatrix} \rightsquigarrow \mathcal{N}(A) = Sp\left(\begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right).$

Example: A plane through the origin in $\mathbb{R}^3$ $\vec{z} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \neq \vec{0}$ **Eqn** $ax + by + cz = 0$

$\mathcal{N}([a\ b\ c])$.