Lecture XVI: §3.4 (cont.) Bases for Subspaces

§1. Spanning Sets:

Recall: $S = \{\vec{v}_1, \ldots, \vec{v}_r\}$ spans a subspace $\mathbb{W}$ of $\mathbb{R}^n$ if $\mathbb{W} = Sp(\vec{v}_1, \ldots, \vec{v}_r)$

Example 1: $\mathbb{W}$ = Row Space of $\left(\begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 5 \\ 3 & 5 & 6 \\ -1 & -1 & -4 \end{bmatrix}\right) = Sp\left(\begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}2\\3\\5\end{bmatrix}, \begin{bmatrix}3\\5\\6\end{bmatrix}, \begin{bmatrix}-1\\-1\\-4\end{bmatrix}\right)$

$= Sp\left(\begin{bmatrix}1\\0\\7\end{bmatrix}, \begin{bmatrix}0\\1\\-3\end{bmatrix}\right)$

Example 2: Check if $\left\{ \underset{\vec{v}_1}{\begin{bmatrix}1\\2\\3\end{bmatrix}}, \underset{\vec{v}_2}{\begin{bmatrix}-1\\0\\-7\end{bmatrix}}, \underset{\vec{v}_3}{\begin{bmatrix}2\\7\\0\end{bmatrix}} \right\}$ spans $\mathbb{R}^3$.

Soln: Want to write any $\vec{v} = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$ as $a\begin{bmatrix}1\\2\\3\end{bmatrix} + b\begin{bmatrix}-1\\0\\-7\end{bmatrix} + c\begin{bmatrix}2\\7\\0\end{bmatrix} = \begin{bmatrix}1 & -1 & 2\\2 & 0 & 7\\3 & -7 & 0\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix} = \vec{v}$

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & x_1 \\ 2 & 0 & 7 & x_2 \\ 3 & -7 & 0 & x_3 \end{array}\right] \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & -1 & 2 & x_1 \\ 0 & 2 & 3 & x_2 - 2x_1 \\ 0 & -4 & -6 & x_3 - 3x_1 \end{array}\right] \xrightarrow[R_2 \to R_2/2]{R_3 \to R_3 + 2R_2} \left[\begin{array}{ccc|c} 1 & -1 & 2 & x_1 \\ 0 & 1 & 3/2 & (x_1 - 2x_2)/2 \\ 0 & 0 & 0 & x_3 + 2x_2 - 7x_1 \end{array}\right]$$

We have a solution in a, b, c if and only if $\boxed{x_3 + 2x_2 - 7x_1 = 0}$

A: The vector don't span $\mathbb{R}^3$. Instead, they span the plane $x_3 + 2x_2 - 7x_1 = 0$

(e.g. $\begin{bmatrix}1\\1\\1\end{bmatrix}$ is not in $Sp(v_1, v_2, v_3)$)

- Better spanning set? $x_3 = -2x_2 + 7x_1 \leadsto \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = x_1\begin{bmatrix}1\\0\\7\end{bmatrix} + x_2\begin{bmatrix}0\\1\\-2\end{bmatrix}$

A: $\mathbb{W} = Sp\left(\begin{bmatrix}1\\0\\7\end{bmatrix}, \begin{bmatrix}0\\1\\-2\end{bmatrix}\right)$

Example 3: Check if $\left\{ \begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}-1\\0\\-7\end{bmatrix}, \begin{bmatrix}2\\8\\1\end{bmatrix} \right\}$ spans $\mathbb{R}^3$

Soln: $\left[\begin{array}{ccc|c} 1 & -1 & 2 & x_1 \\ 2 & 0 & 8 & x_2 \\ 3 & -7 & 1 & x_3 \end{array}\right] \underset{\text{row equiv}}{\sim} \left[\begin{array}{ccc|c} 1 & -1 & 2 & x_1 \\ 0 & 1 & 2 & (x_2 - 2x_1)/2 \\ 0 & 0 & 1 & x_3 + 2x_2 - 7x_1 \end{array}\right] \underset{\text{row equiv}}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{array}\right]$

The system is ALWAYS consistent, no condition on $\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$! So the vectors do span $\mathbb{R}^3$.

Alternative spanning set? $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$.

§2. Best spanning sets = minimal ones!

(S spans $\mathbb{W}$ & $S \setminus \{v_i\}$ doesn't span $\mathbb{W}$ for all v_i in S.)

Example: $\mathbb{W} = Sp\left(\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}-1\\0\\-7\end{bmatrix}, \begin{bmatrix}2\\7\\0\end{bmatrix}\right) = Sp\left(\begin{bmatrix}1\\0\\7\end{bmatrix}, \begin{bmatrix}0\\1\\-2\end{bmatrix}\right) = Sp\left(\begin{bmatrix}-1\\0\\-7\end{bmatrix}, \begin{bmatrix}2\\7\\0\end{bmatrix}\right)$

Q: How to get a minimal spanning set for $\mathbb{W} = Sp(\vec{v}_1, \ldots, \vec{v}_r)$?

A: Use linear dependencies!

$$7\begin{bmatrix}1\\2\\3\end{bmatrix} + 3\begin{bmatrix}-1\\0\\-7\end{bmatrix} + (-2)\begin{bmatrix}2\\7\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} \leadsto \begin{bmatrix}1\\2\\3\end{bmatrix} = -\frac{3}{7}\begin{bmatrix}-1\\0\\-7\end{bmatrix} + \frac{2}{7}\begin{bmatrix}2\\7\\0\end{bmatrix}$$

So we don't need $\begin{bmatrix}1\\2\\3\end{bmatrix}$ to generate $\mathbb{W}$!

- If no relations, we know it's a minimal spanning set! Can't get v_i as l.comb of $S \setminus \{v_i\}$.

ALGORITHM:

INPUT : $S = \{\vec{v}_1, \ldots, \vec{v}_r\}$ a spanning set for $\mathbb{W}$

OUTPUT : S' = subset of the input that is a <u>minimal</u> spanning <u>set</u>.

<u>Step 1</u>: Is S l.i.? $\to$ If YES, OUTPUT S

$\searrow$ If NO, find v_i that is a linear comb. of the remaining vectors

New $S = \{\vec{v}_1, \ldots, \vec{v}_{i-1}, \vec{v}_{i+1}, \ldots, \vec{v}_r\}$

<u>Step 2</u> : Repeat Step 1 for New S, ...

<u>Ex 3</u> (cont) $S = \left\{ \begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}-1\\0\\-7\end{bmatrix}, \begin{bmatrix}2\\8\\1\end{bmatrix} \right\} \rightsquigarrow S_{new} = \left\{ \begin{bmatrix}-1\\0\\-7\end{bmatrix}, \begin{bmatrix}2\\8\\1\end{bmatrix} \right\}$ is l.i.? $\to$ YES $\rightsquigarrow$ <u>Output</u>: S_{new}.

<u>Def</u>: A <u>basis</u> $\mathcal{B}$ of a nonzero subspace $\mathbb{W}$ of $\mathbb{R}^n$ is a minimal spanning set.

Algorithm yields two conditions to check:

$$\begin{cases} (B1) & \mathcal{B} \text{ spans } \mathbb{W} \\ (B2) & \mathcal{B} \text{ is linearly independent} \end{cases}$$

<u>⚠</u>: $\{0\}$ is l.d., so $\mathbb{W} = \{0\}$ has <u>no</u> basis.

<u>Examples</u>: (1) $\left\{ e_1 = \begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}, e_2 = \begin{bmatrix}0\\1\\0\\\vdots\\0\end{bmatrix}, \ldots, e_n = \begin{bmatrix}0\\\vdots\\0\\1\end{bmatrix} \right\}$ <u>canonical basis</u> for $\mathbb{R}^n$

$\begin{bmatrix}x_1\\\vdots\\x_n\end{bmatrix} = x_1 \begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix} + x_2 \begin{bmatrix}0\\1\\\vdots\\0\end{bmatrix} + \cdots + x_n \begin{bmatrix}0\\\vdots\\0\\1\end{bmatrix}$

(2) $\left\{ \begin{bmatrix}1\\0\\7\end{bmatrix}, \begin{bmatrix}0\\1\\-2\end{bmatrix} \right\}$ is a basis for $\{x_3 + 2x_2 - 7x_1 = 0\}$ = plane in $\mathbb{R}^3 = \mathbb{W}$

$\begin{bmatrix}x_1\\x_2\\+7x_1-2x_2\end{bmatrix} = x_1\begin{bmatrix}1\\0\\7\end{bmatrix} + x_2\begin{bmatrix}0\\1\\-2\end{bmatrix}$ $\hookrightarrow$ in $\mathbb{W}$

(3) $\left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix} \right\}$ is a basis for $\mathbb{R}^3$. $\left(\begin{bmatrix}x\\y\\z\end{bmatrix} = z\begin{bmatrix}1\\1\\1\end{bmatrix} + (y-z)\begin{bmatrix}1\\1\\0\end{bmatrix} + (x-y)\begin{bmatrix}1\\0\\0\end{bmatrix} \right)$

$\left[\begin{bmatrix}x\\y\\z\end{bmatrix}\right]_{\mathcal{B}} = \begin{bmatrix}x-y\\y-z\\z\end{bmatrix}$

<u>Theorem</u> = Uniqueness of representation

Pick $\mathbb{V} \neq \{0\}$ a subspace of $\mathbb{R}^n$ with basis $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_r\}$. Then each $\vec{v}$ in $\mathbb{W}$ can be represented in a <u>unique way</u> as linear combination of $\vec{v}_1, \ldots \vec{v}_r$

"Given $\vec{v}$, we can find <u>unique</u> scalars $\alpha_1, \ldots, \alpha_r$ with $\vec{v} = \alpha_1 \vec{v}_1 + \cdots + \alpha_r \vec{v}_r$."

<u>Name</u>: scalars $\alpha_1, \ldots \alpha_r$ = <u>coordinates</u> of $\vec{v}$ <u>with respect to $\mathcal{B}$</u>

Write $[\vec{v}]_{\mathcal{B}} = \begin{bmatrix}\alpha_1\\\vdots\\\alpha_r\end{bmatrix}$ in $\mathbb{R}^r$. (order of $\mathcal{B}$ matters!)

<u>Ex</u>: $x = x_1 \begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix} + x_2 \begin{bmatrix}0\\1\\0\\\vdots\\0\end{bmatrix} + \cdots + x_n \begin{bmatrix}0\\\vdots\\0\\1\end{bmatrix}$ so $[x]_{\mathcal{B}} = \begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix}$ "usual coordinates"

<u>Proof</u>: • Scalars exist because B spans W

• Uniqueness: If $\vec{v} = \alpha_1 \vec{v}_1 + \dots + \alpha_r \vec{v}_r = \beta_1 \vec{v}_1 + \dots + \beta_r \vec{v}_r$,

then $(\alpha_1 - \beta_1)\vec{v}_1 + \dots + (\alpha_r - \beta_r)\vec{v}_r = \vec{0}$

B is l.i forces $\alpha_1 - \beta_1 = \alpha_2 - \beta_2 = \dots = \alpha_r - \beta_r = 0$ so $\alpha_i = \beta_i$ for all i.

§3. Finding a basis from a spanning set $\{\vec{v}_1, \dots, \vec{v}_r\}$

<u>METHOD 1</u>: View W as Range $(\underbrace{[\vec{v}_1 \cdots \vec{v}_r]}_{=A})$ A of size $n \times r$

Find $A \underset{\text{row equiv}}{\sim} A'$ with A' in REF.

<u>Answer</u>: Basis $= \{\vec{v}_i$'s indexed by <u>dependent variables</u> of Null(A) $=$ Null$(A')\}$.

↳ size $=$ rank(A).

<u>Example</u>: $W = \mathrm{Sp}\left(\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ -4 \end{bmatrix} \right)$
$\quad v_1 \quad v_2 \quad v_3 \quad v_4$

$$A = \begin{bmatrix} 1 & 2 & 3 & -1 \\ 2 & 3 & 5 & -1 \\ 1 & 5 & 6 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = A'$$

dep vars $= x_1, x_2$
indep " $= x_3, x_4$

Solns: $\begin{cases} x_1 = -x_3 - x_4 \\ x_2 = -x_3 + x_4 \\ x_3, x_4 \text{ any} \end{cases}$ $\quad \underline{x} = x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ gives 2 dependencies $A\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} = A\begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \vec{0}$

$-\vec{v}_1 - \vec{v}_2 + \vec{v}_3 = \vec{0}$ & $-v_1 + v_2 + v_4 = \vec{0}$

$\rightsquigarrow \vec{v}_3 = \vec{v}_1 + \vec{v}_2$ $\qquad \vec{v}_4 = \vec{v}_1 - \vec{v}_2$

So $B = \{\vec{v}_1, \vec{v}_2\}$ is the basis for W. (• generate ✓ • linearly indep ✓)

<u>METHOD 2</u>: View W as RowSpace $\left(\underbrace{\begin{bmatrix} \vec{v}_1^{\,t} \\ \vdots \\ \vec{v}_r^{\,t} \end{bmatrix}}_{=A}\right)$ A of size $n \times r$

• Use $A \underset{\text{row equiv}}{\sim} A'$ in EF or REF

<u>Answer</u>: Basis $= \{$ nonzero rows of A', transposed $\}$.

<u>Example</u>: $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 5 \\ 3 & 5 & 6 \\ -1 & -1 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = A'$ $\quad$ Basis $= \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix} \right\}$

$\sim \begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix} \rightsquigarrow$ Basis $= \left\{ \begin{bmatrix} 1 \\ 0 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix} \right\}$