

TODAY: Define inner products (generalizing dot product in \mathbb{R}^n) & find basis that are well-behaved with respect to this inner product.

§1. Inner products of subspaces \mathcal{W} of \mathbb{R}^n

Def: An inner product for (a vector space) \mathcal{W} is a function

$$\langle -, - \rangle : \mathcal{W} \times \mathcal{W} \longrightarrow \mathbb{R}$$

assigning $\{u, v\} \longmapsto \langle u, v \rangle$ in \mathbb{R} for u, v in \mathcal{W} , and satisfying for all u, v, w in \mathcal{W} :

(1) $\langle u, u \rangle \geq 0$, $\langle u, u \rangle = 0$ if and only if $u = \mathbf{0}$ ("non-degen. condition")

(2) $\langle u, v \rangle = \langle v, u \rangle$ [Symmetric]

(3) $\langle \alpha u, v \rangle = \alpha \langle u, v \rangle = \langle u, \alpha v \rangle$ for any scalar α .

(4) $\langle u+v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$

Note: By symmetry, we also have:

(4') $\langle u, v+w \rangle = \langle u, v \rangle + \langle u, w \rangle$.

(2) - (4') bilinear form.

Examples: ① $\langle u, v \rangle = u^T \cdot v$ for u, v in \mathbb{R}^n
 $= u \cdot v$ (usual dot product)

(1) becomes $\|u\|^2 = 0$ & $\|u\| = 0$ if & only if $u = \mathbf{0}$.

(2) - (4) follow from matrix multiplication.

② Q symmetric matrix of rank n . Then $\langle u, v \rangle := u^T Q v$.

(2) $\langle v, u \rangle = v^T Q u = \underbrace{(v^T Q u)}_{\in \mathbb{R}} = u^T Q^T (v^T)^T = u^T Q v = \langle u, v \rangle$

(3) - (4) True

(1) not always true. Eg $Q = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ $\mathcal{W} = \mathbb{R}^2$ $[u_1, u_2] \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = u_1^2 - u_2^2$

• If $u_1 = u_2 = 1$ we get $\langle u, u \rangle = 0$ but $\begin{bmatrix} 1 \\ 1 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Obs: Condition that guarantees (1) to hold? \underline{A} : Q "positive definite" ["all eigenvalues are > 0 "]

Obs: An inner product defines a norm on \mathcal{W} : $\|v\| = \sqrt{\langle v, v \rangle}$.

§ 2. Orthogonal bases for \mathbb{R}^n

From now on, we work with the classical inner product in \mathbb{R}^n :

Def: If u, v are vectors in \mathbb{R}^n , we say u & v are orthogonal or perpendicular if $\langle u, v \rangle = u^T \cdot v = 0$. Write $u \perp v$.

Def: A set of vectors $S = \{v_1, \dots, v_p\}$ in \mathbb{R}^n is orthogonal if each pair v_i, v_j with $i \neq j$ is orthogonal.

Example: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is an orthogonal set in \mathbb{R}^3 because $v_1^T v_2 = v_1^T v_3 = v_2^T v_3 = 0$.

• Why do we care? These sets are linearly independent unless they contain $\mathbf{0}$.

Theorem: If S is an orthogonal set of nonzero vectors in \mathbb{R}^n , then S is l.i.

Proof: Write $S = \{\vec{v}_1, \dots, \vec{v}_p\}$ & $a_1 \vec{v}_1 + \dots + a_p \vec{v}_p = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$

$$\text{Then } 0 = \vec{v}_1^T \cdot \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} = v_1^T (a_1 \vec{v}_1 + \dots + a_p \vec{v}_p) = a_1 \underbrace{v_1^T v_1}_{=0} + a_2 \underbrace{v_1^T v_2}_{=0} + \dots + a_p \underbrace{v_1^T v_p}_{=0}$$

Since $\vec{v}_1 \neq \mathbf{0}$, then $\|\vec{v}_1\| \neq 0$ So $0 = a_1 \|\vec{v}_1\|^2$ forces $a_1 = 0$

$$\text{Similarly: } 0 = \vec{v}_2^T \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} = a_1 \underbrace{v_2^T v_1}_{=0} + a_2 \underbrace{v_2^T v_2}_{\neq 0} + \dots + a_p \underbrace{v_2^T v_p}_{=0} = a_2 \|\vec{v}_2\|^2$$

$$\vdots$$
$$0 = \vec{v}_p^T \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} = a_1 \underbrace{v_p^T v_1}_{=0} + a_2 \underbrace{v_p^T v_2}_{=0} + \dots + a_p \underbrace{v_p^T v_p}_{\neq 0} = a_p \|\vec{v}_p\|^2$$

Since $\vec{v}_2, \dots, \vec{v}_p \neq \mathbf{0}$, we conclude $a_1 = a_2 = \dots = a_p = 0$ & so S is l.i. \square

Def: Fix $\mathbb{W} \neq \{\mathbf{0}\}$ subspace of \mathbb{R}^n with basis $B = \{w_1, \dots, w_p\}$

• We say B is an orthogonal basis if B is a basis consisting of orthogonal vectors.

• Orthonormal basis if B is an orthogonal basis &

$$\|w_1\| = \|w_2\| = \dots = \|w_p\| = 1 \quad (\text{all vectors have norm 1}).$$

Example: $\mathbb{W} = \text{span} \left(\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right)$ subspace of \mathbb{R}^3

• $B = \{v_1, v_2\}$ is orthogonal basis: $v_1^T v_2 = 1 - 2 + 1 = 0$. (so l.i.)

• Not orthonormal basis $\| \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \| = \sqrt{1+1+1} = \sqrt{3}$, $\| \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \| = \sqrt{1+1+1} = \sqrt{3}$. L18 [3]

→ We can turn B into an orthonormal basis as follows:

$$B' = \left\{ \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

In general: $B = \{w_1, \dots, w_p\}$ orthogonal vectors $\Rightarrow B' = \left\{ \frac{w_1}{\|w_1\|}, \dots, \frac{w_p}{\|w_p\|} \right\}$ orthonormal basis

• Advantage of orthonormal basis: Computing coordinates is easy! $\hookrightarrow W = \text{Sp}(w_1, \dots, w_p)$

Theorem: Fix $B = \{v_1, \dots, v_p\}$ is an orthonormal basis for $W \neq \{0\}$ subspace of \mathbb{R}^n .

Then, for each v in W : $[v]_B = \begin{bmatrix} v^T v_1 \\ \vdots \\ v^T v_p \end{bmatrix}$

Proof: By definition $[v]_B = \begin{bmatrix} a_1 \\ \vdots \\ a_p \end{bmatrix}$ means $v = a_1 \vec{v}_1 + \dots + a_p \vec{v}_p$.

Then $v^T \cdot v_1 = (a_1 v_1 + \dots + a_p v_p)^T v_1 = a_1 \underbrace{v_1^T v_1}_{=1} + a_2 \underbrace{v_2^T v_1}_{=0} + \dots + a_p \underbrace{v_p^T v_1}_{=0}$

So $v^T \cdot v_1 = a_1$

Similarly for the others: $v^T v_2 = a_1 \cdot 0 + \dots + a_{i-1} \cdot 0 + a_i \underbrace{v_i^T v_i}_{=1} + 0 + \dots + 0 = a_i$

Note: If B is an orthogonal basis $B = \{v_1, \dots, v_p\}$ then:

$$[v]_B = \begin{bmatrix} \frac{v^T v_1}{\|v_1\|^2} \\ \vdots \\ \frac{v^T v_p}{\|v_p\|^2} \end{bmatrix}$$

$$v = a_1 \frac{v_1}{\|v_1\|} + \dots + a_p \frac{v_p}{\|v_p\|}$$

$$\hookrightarrow a_1 = v^T \frac{v_1}{\|v_1\|}, a_2 = v^T \frac{v_2}{\|v_2\|}, \dots, a_p = v^T \frac{v_p}{\|v_p\|}$$

$$\text{So } v = \boxed{\frac{v^T v_1}{\|v_1\|^2}} v_1 + \boxed{\frac{v^T v_2}{\|v_2\|^2}} v_2 + \dots + \boxed{\frac{v^T v_p}{\|v_p\|^2}} v_p$$

§ 3. Gram-Schmidt Algorithm:

• Input: $B = \{w_1, \dots, w_p\}$ basis for a subspace W of \mathbb{R}^n .

• Output: An orthogonal basis for $W = \{u_1, \dots, u_p\}$.

• Routine:

$$u_1 = w_1$$

$$u_2 = w_2 - \frac{u_1^T w_2}{\|u_1\|^2} u_1$$

$$u_3 = w_3 - \frac{u_1^T w_3}{\|u_1\|^2} u_1 - \frac{u_2^T w_3}{\|u_2\|^2} u_2$$

⋮

In general:
for $j=2, \dots, p$

$$u_j = w_j - \frac{u_1^T w_j}{\|u_1\|^2} u_1 - \frac{u_2^T w_j}{\|u_2\|^2} u_2 - \dots - \frac{u_{j-1}^T w_j}{\|u_{j-1}\|^2} u_{j-1}$$

↑
scalars

Q: Why these scalars?

• Assume $\{u_1, \dots, u_{j-1}\}$ is an orthogonal set.

• Need to check: $u_1^T u_j = u_2^T u_j = \dots = u_{j-1}^T u_j = 0$.

Write $u_j = w_j - a_1 u_1 - \dots - a_{j-1} u_{j-1}$
($\{u_1, \dots, u_j\}$ will be l.i.)

T.B.O.

$$u_i^T u_j = u_i^T (w_j - a_1 u_1 - a_2 u_2 - \dots - a_{j-1} u_{j-1})$$

$$= u_i^T w_j - a_1 \frac{u_i^T u_1}{\|u_1\|^2} - a_2 \frac{u_i^T u_2}{\|u_2\|^2} - \dots - a_{j-1} \frac{u_i^T u_{j-1}}{\|u_{j-1}\|^2}$$

So to get $0 = u_i^T u_j$ we must satisfy: $0 = u_i^T w_j - a_i \|u_i\|^2$

$$\Rightarrow a_i = \frac{u_i^T w_j}{\|u_i\|^2}$$

Similarly
 $i=1, \dots, j-1$

$$0 \stackrel{?}{=} u_i^T u_j = u_i^T (w_j - a_1 u_1 - \dots - a_i u_i - \dots - a_{j-1} u_{j-1})$$

$$0 = u_i^T w_j - a_1 \frac{u_i^T u_1}{\|u_1\|^2} - \dots - a_i \frac{u_i^T u_i}{\|u_i\|^2} - \dots - a_{j-1} \frac{u_i^T u_{j-1}}{\|u_{j-1}\|^2}$$

$$0 = u_i^T w_j - a_i \|u_i\|^2$$

$$\text{if \& only if } a_i = \frac{u_i^T w_j}{\|u_i\|^2}$$

for all $j=1, \dots, p$

• By construction $S_p(u_1, \dots, u_j) = S_p(w_1, \dots, w_j)$ so in the end, we set $S_p(u_1, \dots, u_p) = W$ & $\dim W = p$ so $B = \{u_1, \dots, u_p\}$ is a basis

B is an orthogonal set by construction.

Example: $W = \mathbb{R}^3$ with basis $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix} \right\}$ G.S. $\Rightarrow \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix} \right\}$

$$u_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow u_1^T w_2 = 1, u_1^T w_3 = 1, \|u_1\|^2 = 1$$

$$u_2 = w_2 - \frac{u_1^T w_2}{\|u_1\|^2} u_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} - \frac{1}{1} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

$$\Rightarrow u_2^T w_3 = 1, u_2^T w_3 = -6, \|u_2\|^2 = 4$$

$$u_3 = w_3 - \frac{u_1^T w_3}{\|u_1\|^2} u_1 - \frac{u_2^T w_3}{\|u_2\|^2} u_2 = \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix} - \frac{1}{1} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \frac{-6}{4} \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}$$