Lecture XIX : §5.1-5.2 Vector Spaces

§1. Motivation:

So far: constructed $\mathbb{R}^n$ & vector subspaces of $\mathbb{R}^n$:

(1) $\mathbb{W} = \mathbb{R}^n$

(2) $\mathbb{W} = \{\mathbb{0}\}$

(3) $\mathbb{W} = Sp(\vec{v}_1, \ldots, \vec{v}_r)$

(4) = $\mathbb{W} = \mathcal{N}(\underset{m\times n}{A}) = \left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} : A \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \text{ in } \mathbb{R}^m \right\}$ Nullity Space

(5) $\mathbb{W}$ = Range(A) = Column Space $(A) = Sp(a_1, \ldots, a_n)$ subspace of $\mathbb{R}^m$. $\quad \underset{m\times n}{A = [a_1 \cdots a_n]}$

(6) $\mathbb{W}$ = Row Space (A) = Column Space (A^T) subspace of $\mathbb{R}^n$.

Common features:
- $\mathbb{0}$ in $\mathbb{W}$
- We can add two vectors of $\mathbb{W}$ (using + in $\mathbb{R}^n$) & remain in $\mathbb{W}$
- —— scalar multiply a vector in $\mathbb{W}$ with any scalar (using the usual scalar product operation in $\mathbb{R}^n$) & remain in $\mathbb{W}$.
- + & scalar mult. have nice algebraic properties (inherited from $\mathbb{R}^n$)

We see these properties in other settings:

Example 1: A set of solutions to the differential equation $y''_{(x)} + y_{(x)} = 0$

- Solution 1: $y(x) = e^x$
- Solution 2: $y(x) = -e^{-x}$
- Any linear combination of these $y(x) = c_1 e^x + c_2(-e^{-x})$ is also a solution.

$$y'(x) = c_1 e^x + c_2 e^{-x}$$
$$y''(x) = c_1 e^x + c_2(-e^{-x}) = y(x) \checkmark$$

- Claim: These 2 solutions are "linearly independent":

$$0 = c_1 e^x + c_2(-e^{-x}) \quad \text{as functions}$$

We want to conclude that $c_1 = c_2 = 0$.

We do so by evaluating at suitable values of x:

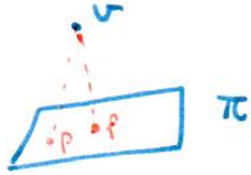
- At $x = 0$: $\quad 0 = c_1 e^0 + c_2(-e^{-0}) = c_1 - c_2$, so $c_2 = c_1$

- At $x=1$ $\quad 0 = c_1 e + c_1(-e^{-1}) = c_1\underbrace{\left(e - \frac{1}{e}\right)}_{\neq 0}$ so $c_1 = 0$

Conclude: $c_1 = c_2 = 0$.

- Why do we care? We can show $\{y \mid y'' + y = 0\} = Sp(e^x, -e^{-x})$

Example 2: Fix



π plane through $(0,0,0)$ in $\mathbb{R}^3$

Q: What is the closest point p to v in the plane π?

A: Need to minimize $d_{(v,p)} = \|v - p\|$ for p in π
$$\|v-p\| = \sqrt{\langle v-p, v-p\rangle}$$

Answer $\overline{vp} \perp \pi$ so it must be proportional to $\vec{n}$ = normal to π.

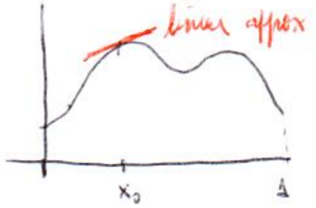
Similarly, fix the space V of all continuous functions over $[0,1]$

We have $P_2 = \{a + bx + cx^2 ; a, b, c \text{ in } \mathbb{R}\}$ (polynomials of degree ≤ 2)

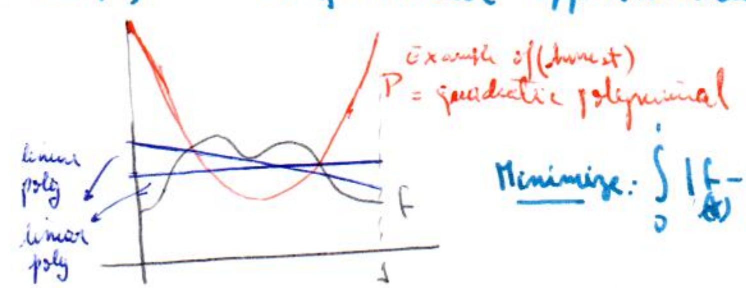
- Want to find $P = a + bx + cx^2$ minimizing $\|f - P\| = \sqrt{\langle f-P, f-P\rangle}$, i.e. distance from f to P_2.

We take $\langle g, h\rangle = \int_0^1 g_{(x)} h_{(x)}\, dx$ (inner product for cont functions on $[0,1]$)

This extends the notion of linear approximation for a ~~continuous~~ & differentiable function near a point x_0 (use tangent line / tangent plane to the graph of the function at the point $(x_0, f(x_0))$) to a quadratic approximation over the interval $[0,1]$



vs



Minimize: $\int_0^1 |f_{(x)} - P_{(x)}|^2\, dx$

In this case we identify $a + bx + cx^2$ with $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ (vector of coefficients), but we don't have the ~~usual~~ distance from $\mathbb{R}^3$ (not dot product). Declare $\left\langle \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \begin{bmatrix} a' \\ b' \\ c' \end{bmatrix} \right\rangle = \langle a + bx + cx^2, a' + b'x + c'x^2 \rangle$

§2. Vector Spaces:

A general vector space consists of a set of elements (called vectors) $\mathbb{V}$ & a set of scalars S ($\mathbb{R}$ or $\mathbb{C}$) with 2 operations:

(1) <u>Addition</u>: $+ \quad \mathbb{V} \times \mathbb{V} \longrightarrow \mathbb{V}$

$$(u, v) \longmapsto u+v$$

(2) <u>Scalar Multiplication</u> For $\mathbb{R}$: $\bullet : \mathbb{R} \times \mathbb{V} \longrightarrow \mathbb{V}$

$$(\alpha, v) \longmapsto \alpha v$$

Want $+$ & $\bullet$ to have nice algebraic properties.

Examples:
- $\mathbb{R}^n$ & all subspaces discussed on page 1
- $\mathbb{R}^{n \times m}$ = matrices of size $n \times m$ with usual $+$ & scalar mult.
- Solutions to homogeneous differential equations (e.g. $y'' + y = 0$)

<u>Formal definition</u>: A set of elements $\mathbb{V}$ is a <u>vector space over $\mathbb{R}$</u> if we can define addition & scalar multiplication on $\mathbb{V}$ & the following properties hold for any u, v, w in $\mathbb{V}$ & scalars α, β in $\mathbb{R}$:

- <u>Closure properties</u>:
  - (C1) $u+v$ is a vector in $\mathbb{V}$ for u, v in $\mathbb{V}$
  - (C2) αu ——————— for u in $\mathbb{V}$, α in $\mathbb{R}$
- <u>Properties of Addition</u>:
  - (A1) $u+v = v+u$ [Commutative]
  - (A2) $u+(v+w) = (u+v)+w$ [Associative]
  - (A3) There is a vector $\mathbb{0}$ in $\mathbb{V}$ such that $v + \mathbb{0} = \mathbb{0} + v = v$ for all v in $\mathbb{V}$
    $\mathbb{0}$ = Neutral Element for $\mathbb{V}$
  - (A4) Given v in $\mathbb{V}$ there is w in $\mathbb{V}$ such that $v+w = w+v = \mathbb{0}$
    w = Additive inverse for v
- <u>Properties of Scalar Multiplication</u>:
  - (M1) $\alpha \cdot (\beta \cdot v) = (\alpha\beta) \cdot v$ [Commutative]
  - (M2) $\alpha \cdot (u+v) = \alpha \cdot u + \alpha \cdot v$ [Distributive 1]
  - (M3) $(\alpha + \beta) \cdot u = \alpha \cdot u + \beta \cdot u$ [Distributive 2]

(M4) $1 \cdot v = v$ for all v

Example 1: $\mathbb{V} = \{ \mathbb{0} \}$ with $\mathbb{0} + \mathbb{0} = \mathbb{0}$, $\alpha \cdot \mathbb{0} = \mathbb{0}$ satisfies all 10 properties

Example 2: $m \times n$ matrices, $\mathbb{0} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ \vdots & & & & \\ 0 & \cdots & & \cdots & 0 \end{bmatrix}$ with usual $+$ & $\cdot$.

Example 3: $\mathbb{V} = \mathbb{R}^2$ with funky addition $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \oplus \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} u_1+v_1-1 \\ u_2+v_2-1 \end{bmatrix}$

& usual scalar multiplication

- (C1), (C2) holds
- (A1) holds
- (A2) $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \oplus \left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \oplus \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \right) = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \oplus \begin{bmatrix} v_1+w_1-1 \\ v_2+w_2-1 \end{bmatrix} = \begin{bmatrix} u_1+v_1+w_1-2 \\ u_2+v_2+w_2-2 \end{bmatrix}$

$$\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \oplus \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right) \oplus \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} u_1+v_1-1 \\ u_2+v_2-1 \end{bmatrix} \oplus \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} u_1+v_1+w_1-2 \\ u_2+v_2+w_2-2 \end{bmatrix}$$

- (A3) $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \oplus \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} v_1+u_1-1 \\ v_2+u_2-1 \end{bmatrix} \rightsquigarrow \begin{cases} v_1 = v_1+u_1-1 \\ v_2 = v_2+u_2-1 \end{cases} \rightsquigarrow u_1 = u_2 = 1$

Neutral Element $\mathbb{0}_{new} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

- (A4) $\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \oplus \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} v_1+w_1-1 \\ v_2+w_2-1 \end{bmatrix} \rightsquigarrow \begin{matrix} v_1+w_1-1=1 \\ v_2+w_2-1=1 \end{matrix}$ so $\begin{matrix} w_1 = 2-v_1 \\ w_2 = 2-v_2 \end{matrix}$

So inverse for $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 2-v_1 \\ 2-v_2 \end{bmatrix}$

- (M1) holds
- (M2) $\alpha \left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \oplus \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right) = \alpha \begin{bmatrix} u_1+v_1-1 \\ u_2+v_2-1 \end{bmatrix} = \begin{bmatrix} \alpha u_1+\alpha v_1-\alpha \\ \alpha u_2+\alpha v_2-\alpha \end{bmatrix}$

$$\alpha \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \oplus \alpha \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \alpha u_1 \\ \alpha u_2 \end{bmatrix} \oplus \begin{bmatrix} \alpha v_1 \\ \alpha v_2 \end{bmatrix} = \begin{bmatrix} \alpha u_1+\alpha v_1-1 \\ \alpha u_2+\alpha v_2-1 \end{bmatrix}$$

To get $=$ we must have $\alpha = 1$ But scalars are arbitrary!

Conclude: (M2) fails.

(M3) similarly fails

(M4) holds

Conclude: $(\mathbb{R}^2, \oplus, \cdot)$ is NOT a vector space.

Example 4: $\mathbb{V} = \{ 2\times 2$ singular matrices $\}$ is NOT a vector space

(M1) — (M4), (A1), (A2) hold

$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is singular & its $\mathbb{0}$, so (A3) holds

(A4) A singular, Additive inverse $= -A$ is also singular.

(C2) holds A singular, then αA also singular

$\{col_1 A, col_2 A\}$ l.d. $\quad \{\alpha\, col_1 A, \alpha\, col_2 A\}$ also l.d.

(C1) fails: $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ so W is not closed under $+$.

sing $\quad$ sing $\quad$ non-sing

Example 5: $P_2 = \{ a + bx + cx^2 : a, b, c \text{ in } \mathbb{R} \}$ = polynomials of degree ≤ 2

$+ : (a + bx + cx^2) + (a' + b'x + c'x^2) = (a+a') + (b+b')x + (c+c')x^2$

$\cdot \quad \alpha(a + bx + cx^2) = (\alpha a) + (\alpha b)x + (\alpha c)x^2$

. (C1), (C2) hold, (A1), (A2), (M1) — (M4) hold

(A3) $\mathbb{O} = 0 + 0 \cdot x + 0 \cdot x^2$ neutral element

(A4) $-P = (-1)P$ additive inverse

$P_2 = Sp(1, x, x^2)$