

# Lecture XXI: §5.4 Linear Independence & Bases

## 30. Review:

Last time: • Defined  $\mathbb{W}$  subspace of an abstract vector space  $\mathbb{V}$

- (s1) If  $u, v$  in  $\mathbb{W}$ , then  $u+v$  also in  $\mathbb{W}$
- (s2) If  $u$  in  $\mathbb{W}$  &  $\alpha$  in  $\mathbb{R}$  then  $\alpha u$  also in  $\mathbb{W}$
- (s3)  $\mathbf{0}$  in  $\mathbb{W}$

Main example:  $\text{Sp}(v_1, \dots, v_r) = \{ \alpha_1 v_1 + \dots + \alpha_r v_r : \alpha_1, \dots, \alpha_r \text{ in } \mathbb{R} \}$

•  $\mathbb{C}[0, 1]$  has no finite spanning sets.

Example 1  $\{1, x, x^2\}$  spans  $\mathcal{P}_2 = \{a + bx + cx^2 : a, b, c \text{ in } \mathbb{R}\}$

Example 2  $\mathbb{W} = \{ A \text{ in } M_{3 \times 3} : A^T = A \}$  We build a spanning set

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$A^T = \begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}$$

This gives:  $a_{12} = a_{21}$   
 $a_{13} = a_{31}$   
 $a_{23} = a_{32}$

Take  $a_{11}, a_{12}, a_{13}, a_{22}, a_{23}, a_{33}$  out  $= E_{12} + E_{21}$   $= E_{13} + E_{31}$

$$\text{So } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} \stackrel{\downarrow}{=} a_{11} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + a_{12} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + a_{13} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \\ + a_{22} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + a_{23} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} + a_{33} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Conclude:  $\mathbb{W} = \text{Sp}(E_{11}, E_{12} + E_{21}, E_{13} + E_{31}, E_{22}, E_{23} + E_{32}, E_{33})$

## §1. Linear independence:

Def: Fix a vector space  $\mathbb{V}$  &  $v_1, \dots, v_r$  vectors in  $\mathbb{V}$ . We write:

$$(*) \quad \mathbf{0} = \alpha_1 v_1 + \dots + \alpha_r v_r$$

We know  $\alpha_1 = \dots = \alpha_r = 0$  is a solution.

- The set  $\{v_1, \dots, v_r\}$  is linearly independent if the only solution to (\*) is the trivial one,  $\alpha_1 = \alpha_2 = \dots = \alpha_r = 0$
- Otherwise, we say  $\{v_1, \dots, v_r\}$  is linearly dependent.

Obs: This is the same definition as for  $\mathbb{R}^n$ . We can decide l.i vs l.d with the same methods!

METHOD 2: Subsets of  $M_{n \times n}$ 

①  $\{E_{11}, E_{12} + E_{21}, E_{13} + E_{31}, E_{22}, E_{23} + E_{32}, E_{33}\}$  is l.i.

Why? Write  $\mathbf{0} = a_{11}E_{11} + a_{12}(E_{12} + E_{21}) + a_{13}(E_{13} + E_{31}) + a_{22}E_{22} + a_{23}(E_{23} + E_{32}) + a_{33}E_{33}$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} \quad \leftarrow \text{under the "splitting" done in page 2.}$$

Includes all entries agree, so  $a_{11} = a_{12} = a_{13} = a_{22} = a_{23} = a_{33} = 0$  is the unique solution to the equation (\*).

②  $\{v_1 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, v_2 = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix}, v_3 = \begin{bmatrix} 3 & 0 & 5 \\ 0 & 3 & 0 \end{bmatrix}, v_4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\}$  is l.d.

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{0} = a \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} + b \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix} + c \begin{bmatrix} 3 & 0 & 5 \\ 0 & 3 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} a+b+3c & 0 & a+2b+5c+d \\ 0 & a+b+3c & 0 \end{bmatrix}$$

$$\text{We set } \begin{cases} a+b+3c=0 \\ a+2b+5c+d=0 \\ a+b+3c=0 \end{cases} \equiv \begin{cases} a+b+3c=0 \\ a+2b+5c+d=0 \end{cases}$$

$\left. \begin{array}{l} 2 \text{ equations} \\ 4 \text{ unknowns} \\ \text{homogeneous} \end{array} \right\} \Rightarrow \geq 2 \text{ independent variables}$

So we have infinitely many solutions, & so  $\{v_1, v_2, v_3, v_4\}$  is l.d.

Explicit solutions:

$$\begin{bmatrix} 1 & 1 & 3 & 0 \\ 1 & 2 & 5 & 1 \end{bmatrix} \xrightarrow{R_2 \rightarrow R_2 - R_1} \begin{bmatrix} 1 & 1 & 3 & 0 \\ 0 & 1 & 2 & 1 \end{bmatrix} \xrightarrow{R_1 \rightarrow R_1 - R_2} \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 2 & 1 \end{bmatrix} \quad \text{REF}$$

$$\text{so } \begin{cases} a = -c + d \\ b = -2c - d \end{cases}$$

$$\text{Solns: } \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} -c+d \\ -2c-d \\ c \\ d \end{bmatrix} = c \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + d \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\text{We set } -1v_1 - 2v_2 + v_3 = \mathbf{0}$$

$$v_1 - v_2 + v_4 = \mathbf{0}$$

2 "generating" linear relations

$$\Rightarrow \text{Sp}(v_1, v_2, v_3, v_4) = \text{Sp}(v_1, v_2)$$

because  $v_3 = v_1 + 2v_2$   
 $v_4 = -v_1 + v_2$

METHOD 2: For subspaces of functions

①  $\{1, x, x^2\}$  is l.i in  $\mathbb{P}_2$ :  $0 = a + bx + cx^2$

We evaluate at convenient values for  $x$ :

$$\left. \begin{array}{l} x=0 : 0 = a \\ x=1 : 0 = a+b+c \\ x=-1 : 0 = a-b+c \end{array} \right\} \begin{array}{l} = b+c \\ = -b+c \end{array} \quad \left. \begin{array}{l} a=0 \\ b=c=0 \end{array} \right\} \text{ is only solution}$$

②  $\{1, (x+1)^2, (x-1)^2, x^2\}$  is l.d.

$0 = a + b(x+1)^2 + c(x-1)^2 + dx^2$

Option 1: Open up squares, regroup scalars & use  $\{1, x, x^2\}$  is l.i

$0 = a + b(x^2 + 2x + 1) + c(x^2 - 2x + 1) + dx^2$

$0 = (a+b+c) + (2b-2c)x + (b+c+d)x^2$

$$\text{So } \begin{cases} a+b+c=0 & \Rightarrow a+2b=0 \Rightarrow a=-2b \\ 2b-2c=0 & \Rightarrow b=c \\ b+c+d=0 & \Rightarrow 2b+d=0 \Rightarrow d=-2b \end{cases}$$

So  $0 = b(-2 + (x+1)^2 + (x-1)^2 - 2x^2)$  for all  $b$

$0 = -2 + (x+1)^2 + (x-1)^2 - 2x^2$  . minimal relation.

Option 2: Evaluate at convenient values of  $x$ : (pick values where some factors vanish)

At  $x=0$  :  $a + b + c = 0$

At  $x=1$  :  $a + 4b + d = 0$

At  $x=-1$  :  $a + 4c + d = 0$

At  $x=2$  :  $a + 9b + c + 4d = 0$

Solve the system for  $a, b, c, d$

$$\left[ \begin{array}{cccc|c} 1 & 1 & 1 & 0 & 0 \\ 1 & 4 & 0 & 1 & 0 \\ 1 & 0 & 4 & 1 & 0 \\ 1 & 9 & 1 & 4 & 0 \end{array} \right] \xrightarrow{\substack{R_2 \rightarrow R_2 - R_1 \\ R_3 \rightarrow R_3 - R_1 \\ R_4 \rightarrow R_4 - R_1}} \left[ \begin{array}{cccc|c} 1 & 1 & 1 & 0 & 0 \\ 0 & 3 & -1 & 1 & 0 \\ 0 & -1 & 3 & 1 & 0 \\ 0 & 8 & 0 & 4 & 0 \end{array} \right] \xrightarrow{\substack{R_3 \rightarrow \frac{2}{3}R_3 \\ R_4 \rightarrow \frac{2}{3}R_4 \\ R_3 \leftrightarrow R_2}} \left[ \begin{array}{cccc|c} 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 3 & 1 & 0 \\ 0 & 3 & -1 & 1 & 0 \\ 0 & 2 & 0 & 1 & 0 \end{array} \right]$$

$$\xrightarrow{\substack{R_3 \rightarrow R_3 - 3R_2 \\ R_4 \rightarrow R_4 - 2R_2}} \left[ \begin{array}{cccc|c} 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 3 & 1 & 0 \\ 0 & 0 & 8 & 4 & 0 \\ 0 & 0 & 6 & 3 & 0 \end{array} \right] \xrightarrow{\substack{R_3 \rightarrow \frac{R_3}{8} \\ R_4 \rightarrow R_4 - 3R_3}} \left[ \begin{array}{cccc|c} 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 3 & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right] \xrightarrow{\substack{R_2 \rightarrow R_2 + 3R_3 \\ R_4 \rightarrow R_4 - R_3}} \left[ \begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

$\Rightarrow a = d, b = c = -\frac{1}{2}d$   $\Rightarrow 0 = (1 - \frac{1}{2}(1+x)^2 - \frac{1}{2}(x-1)^2 + x^2)$  Relation

Option 3: Take derivatives to get relations:

$$0 = a + b(x+1)^2 + c(x-1)^2 + dx^2 \quad (1)$$

$$\frac{d}{dx} 0 = 2b(x+1) + 2c(x-1) + 2dX = 2(b+c+d)X + 2(b-c) \quad (2)$$

$$\frac{d}{dx} 0 = 2b + 2c + 2d = 2(b+c+d) \quad (3)$$

Take (2): get  $b+c+d=0 \implies d=-2b$   
 $b-c=0 \implies b=c$

Back to (1):  $0 = a + b((x+1)^2 + (x-1)^2) - 2bX^2$

$$0 = a + b((x+1)^2 + (x-1)^2 - 2X^2)$$

$$0 = a + b(2) \quad \text{so } a = -2b.$$

Conclude:  $a = d = -2b, c = b$

Relation:  
( $\implies b=1$ )

$$0 = -2 + (x+1)^2 + (x-1)^2 - 2X^2$$

(double of relation obtained using Option 2)

### §3. Bases for abstract vector spaces:

Def: Let  $\mathbb{V}$  be a vector spaces. A set  $B = \{v_1, \dots, v_r\}$  is a basis for  $\mathbb{V}$  if

(1)  $B$  is a spanning set for  $\mathbb{V}$

(2)  $B$  is linearly independent (equivalently:  $B$  is a minimal spanning set)

Examples:  $M_{2 \times 3}$  has basis  $\{E_{ij} : \begin{matrix} 1 \leq i \leq 2 \\ 1 \leq j \leq 3 \end{matrix}\}$  (6 elements)

$M_{m \times n}$  has basis  $\{E_{ij} : \begin{matrix} 1 \leq i \leq m \\ 1 \leq j \leq n \end{matrix}\}$  ( $m \cdot n$  elements)

$\mathcal{P}_d = \{a_0 + a_1x + \dots + a_dx^d\}$  has basis  $\{1, x, x^2, \dots, x^d\}$  ( $d+1$  elements)  
 $a_0, a_1, \dots, a_d$  in  $\mathbb{R}$

Upshot: Bases of a vector space  $\mathbb{V}$  with a finite spanning set all have the same number of element. We define this number as the dimension of  $\mathbb{V}$

ALGORITHM: Start from a spanning set  $S = \{v_1, \dots, v_r\}$  for  $\mathbb{V}$

(1) If  $S$  is L.I., then  $S$  is a basis

(2)  $\text{---}$  L.D., use a relation to write some  $v_i$  as a linear comb. of the others, Set  $S' = \{v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_r\} \begin{cases} \text{---} \text{ L.I., done} \\ \text{---} \text{ L.D., ...} \end{cases}$

Example:  $S = \{1, (x+1)^2, (x-1)^2, x\}$  &  $W = \mathcal{P}_2$

415

•  $S$  spans  $\mathcal{P}_2$ :

$$\begin{cases} 1 \in \text{Sp}(S) \\ x \in \text{Sp}(S) \\ x^2 = (x+1)^2 - 2x - 1 \in \text{Sp}(S) \end{cases}$$

since a basis for  $\mathcal{P}_2$  can be generated from  $S$ , then  $S$  is a spanning set.

•  $S$  is L.D.:

$$0 = a + b(x+1)^2 + c(x-1)^2 + dx$$

$$\frac{d}{dx} 0 = 2b(x+1) + 2c(x-1) + d = 2(b+c)x + 2(b-c) + d = 2(+2b) + d = +4b + d$$

$$\frac{d}{dx} 0 = 2(b+c) \quad \Rightarrow \quad b+c=0 \quad \Rightarrow \quad -b=c$$

$$\uparrow \quad \text{so } \boxed{d = -4b}$$

Replace in original equation:

$$0 = a + b(x+1)^2 - b(x-1)^2 - 4bx$$

$$0 = a + b \underbrace{(x+1)^2 - (x-1)^2 - 4x}_{=0} \quad \text{so } a=0$$

Conclude there is a unique relation (set  $b=1$ )

$$0 = (x+1)^2 - (x-1)^2 - 4x$$

$$\text{So } (x+1)^2 = (x-1)^2 + 4x \quad \Rightarrow \quad S' = \{1, (x-1)^2, x\} \text{ spans & li.}$$

$$\text{So } (x-1)^2 = -(x+1)^2 - 4x \quad \Rightarrow \quad S_2 = \{1, (x+1)^2, x\} \text{ is a basis}$$

$$x = \frac{1}{4}(x+1)^2 - \frac{1}{4}(x-1)^2 \quad \Rightarrow \quad S_3 = \{1, (x+1)^2, (x-1)^2\} \text{ is a basis}$$

Note: I can remove any vector from  $S$  except 1 & get basis for  $\mathcal{P}_2$