Lecture XXII: §5.4 (cont.) Coordinates in abstract vector spaces

§1 Bases:

$\mathbb{V} \neq \{0\}$ a fixed vector space. A set $\mathcal{B} = \{v_1, \ldots, v_r\}$ is a **basis** for $\mathbb{V}$ if

(1) $\mathcal{B}$ is a spanning set for $\mathbb{V}$

(2) $\mathcal{B}$ is l.i. (equivalently: minimal spanning set)

Examples
(0) $\mathbb{R}^n$ basis $\mathcal{B} = \{e_1, \ldots, e_n\}$ canonical basis $\left(e_i = \begin{cases} 1 & \text{in } i^{th} \text{ position} \\ 0 & \text{elsewhere} \end{cases}\right)$

(1) $\mathcal{P}_d = \{a_0 + a_1 x + a_2 x^2 + \cdots + a_d x^d : a_0, \ldots, a_d \text{ in } \mathbb{R}\}$

Basis : $\{1, x, x^2, \ldots, x^d\}$ ($d+1$ elements)

(2) $M_{m\times n} = \{A = (a_{ij})_{\substack{1\leq i\leq m \\ 1\leq j\leq n}} : a_{ij} \text{ in } \mathbb{R}\}$

Basis $= \{E_{ij} : \substack{1\leq i\leq m \\ 1\leq j\leq n}\}$ ($m\cdot n$ elements) $\qquad E_{ij} = m\times n$ matrix with $\begin{cases} 1 & \text{in } (i,j) \text{ entry} \\ 0 & \text{all other entries} \end{cases}$

⇝ Same algorithm as in subspaces of $\mathbb{R}^n$ to build a basis from a spanning set.

§2. Coordinates for vector spaces with bases:

Basis for $\mathbb{V}$ = coordinate system for $\mathbb{V}$ ⇝ use it to work with $\mathbb{V}$ (compute ~~sub~~spaces, dim, ...)

Ex 1 $\mathbb{R}^n$ has coordinate system $\mathcal{B} = \{e_1, \ldots, e_n\}$

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \underline{\underline{x_1}} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \underline{\underline{x_2}} \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \cdots + \underline{\underline{x_n}} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$\left[\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}\right]_{\mathcal{B}} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \quad \text{means} \quad \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1 e_1 + \cdots + x_n e_n$$
(order for $\mathcal{B}$ matters!)

Ex 2: $M_{m\times n}$ has coordinate system $\mathcal{B} = \{E_{ij}\}$

$$A = (a_{ij})_{i,j} = \sum_{i,j} a_{ij} E_{ij}$$

Ex $M_{2\times 3}$: $[A]_{\mathcal{B}} = \begin{bmatrix} a_{11} \\ a_{12} \\ a_{13} \\ a_{21} \\ a_{22} \\ a_{23} \end{bmatrix}$ in $\mathbb{R}^6$ means $A = a_{11}E_{11} + a_{12}E_{12} + a_{13}E_{13} + a_{21}E_{21} + a_{22}E_{22} + a_{23}E_{23}$

Here: $6 = |\mathcal{B}|$

- Upshot: after taking coordinates with respect to $\mathcal{B}$, we can identify $\mathbb{V}$ with $\mathbb{R}^{|\mathcal{B}|}$
- More precisely:

Theorem 1: Given a vector space $\mathbb{W}$ with basis $\mathcal{B} = \{v_1, \dots v_p\}$, the representation of each v in $\mathbb{W}$ as a linear combination of $\mathcal{B}$, meaning the scalars in

$$(*) \qquad v = \underline{\underline{\alpha_1}}\, v_1 + \cdots + \underline{\underline{\alpha_p}}\, v_p$$

are unique. We call them the coordinates of v with respect to $\mathcal{B}$ and write $[v]_{\mathcal{B}} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_p \end{bmatrix}$ = vector in $\mathbb{R}^p$

⚠ Order of $\mathcal{B}$ matters!

Proof: • We have an expression like $(*)$ because $\{v_1, \dots v_p\}$ spans $\mathbb{W}$.

• Assume we have 2 solutions to $(*)$ & take their difference:

$$\begin{array}{rcl} v &=& \alpha_1 v_1 + \cdots + \alpha_p v_p \\ - \quad v &=& \beta_1 v_1 + \cdots + \beta_p v_p \\ \hline \mathbb{O} &=& (\alpha_1 - \beta_1) v_1 + \cdots + (\alpha_p - \beta_p) v_p \end{array}$$

Since $\{v_1, \dots v_p\}$ is l.i., we must have $\begin{cases} \alpha_1 - \beta_1 = 0 \\ \vdots \\ \alpha_p - \beta_p = 0 \end{cases}$ so $\begin{cases} \alpha_1 = \beta_1 \\ \vdots \\ \alpha_p = \beta_p \end{cases}$ ⇝ We get that there is a unique solution to $(*)$.

Consequence: We can identify $\mathbb{W}$ with $\mathbb{R}^p$ via $v \longleftrightarrow [v]_{\mathcal{B}}$. That is, we have a bijection $\Psi: \mathbb{W} \longrightarrow \mathbb{R}^p$ (1-to-1 map) that behaves well with respect to the linear structure of the two vector spaces (see Lemma in page 3)

Meaning
$$\Psi(v + w) = \Psi(v) + \Psi(w) \quad \text{for } v, w \text{ in } \mathbb{W}$$
$$\Psi(\alpha v) = \alpha \Psi(v) \quad \text{for } v \text{ in } \mathbb{W} \text{ \& } \alpha \text{ in } \mathbb{R}$$

These properties define linear Transformations between vector spaces (§3.7 & §5.7)

In this case Ψ is a lin. transf that is also bijective.

Example: (1) $\left[\begin{pmatrix} a & 0 & b \\ 0 & c & 0 \end{pmatrix}\right]_{\{E_{ij}\}} = \begin{bmatrix} a \\ 0 \\ b \\ 0 \\ c \\ 0 \end{bmatrix}$ in $\mathbb{R}^6$
in $M_{2\times 3}$

(2) $\mathbb{W} = Sp\langle \underbrace{E_{11}, E_{13}, E_{22}}_{=: \mathcal{B} \text{ basis for } \mathbb{W}} \rangle$ ⇝ $\left[\begin{pmatrix} a & 0 & b \\ 0 & c & 0 \end{pmatrix}\right]_{\mathcal{B}} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ in $\mathbb{R}^3$

⇝ $\mathbb{W} \longleftrightarrow \mathbb{R}^3$.

(3) $\mathbb{W} = \mathcal{P}_2$: $\quad [a+bx+cx^2]_{\{1,x,x^2\}} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

$\mathbb{W} = \mathcal{P}_2$ in $\mathbb{V} = \mathcal{P}_3$ $\quad [a+bx+cx^2]_{\{1,x,x^2,x^3\}} = \begin{bmatrix} a \\ b \\ c \\ 0 \end{bmatrix} \to \mathbb{W} \equiv \{x_4 = 0 \text{ in } \mathbb{R}^4\}$

Lemma: Identification via coordinates with respect to a fixed basis B behaves well with respect to addition & scalar multiplication in $\mathbb{V} = Sp(B)$.

(1) $[\mathbb{O}]_B = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ in $\mathbb{R}^p$ $\quad (p = |B|)$

(2) $[v+w]_B = [v]_B + [w]_B$ for any v, w in $\mathbb{V}$

(3) $[\alpha w]_B = \alpha [v]_B$ for v in $\mathbb{V}$, α in $\mathbb{R}$.

Q What else can we do? Decide l i, spanning, ...

Theorem 2: Fix $\mathbb{V}$ vector space with basis $B = \{v_1, \dots, v_p\}$ & a set $S = \{w_1, \dots w_r\}$ of vectors in $\mathbb{V}$. Write $T = \{[w_1]_B, \dots [w_r]_B\}$ in $\mathbb{R}^p$.

(1) A vector v from $\mathbb{V}$ lies in $Sp(S)$ if and only if $[v]_B$ lies in $Sp(T)$

(2) The set S is linearly independent ——— T is l i in $\mathbb{R}^p$

Proof: (1) $v = a_1 w_1 + \dots + a_r w_r$ if and only if $[v]_B = a_1 [w_1]_B + \dots + a_r [w_r]_B$ (By Lemma). Note: we use the exact same scalars!

(2) Follows from proof of (1). Scalars are forced to be 0 for $\mathbb{O}$ in $\mathbb{V}$ if and only if they are forced to be 0 for $\mathbb{O}$ in $\mathbb{R}^p$.

Consequence 1: S is a basis for $\mathbb{V}$ if and only if T is a basis for $\mathbb{R}^p$.
In particular, all basis for $\mathbb{V}$ have the same number of vectors.
We call this number the <u>dimension</u> of $\mathbb{V}$

Consequence 2: Fix $\mathbb{V}$ vector space of dimension p. Then:

(1) A set of $p+1$ or more vectors in $\mathbb{V}$ is linearly dependent.

(2) Any set of $p-1$ or fewer ——— cannot span $\mathbb{V}$.

(3) Any set of p linearly independent vectors in $\mathbb{V}$ is a basis for $\mathbb{V}$.

(4) ——— vectors in $\mathbb{V}$ that spans $\mathbb{V}$ is a basis for $\mathbb{V}$.

Example: $S = \{ \overset{v_1}{1}, \overset{v_2}{(x+1)^2}, \overset{v_3}{(x-1)^2}, \overset{v_4}{x} \}$ in $\mathcal{P}_2$, $\dim \mathcal{P}_2 = 3$. $B = \{1, x, x^2\}$ in $\mathcal{P}_2$ $\updownarrow$ $\{e_1, e_2, e_3\}$ in $\mathbb{R}^3$

$$[1]_B = \begin{bmatrix}1\\0\\0\end{bmatrix} \qquad [(x+1)^2]_B = [x^2+2x+1]_B = \begin{bmatrix}1\\2\\1\end{bmatrix}$$

$$[(x-1)^2]_B = [x^2-2x+1]_B = \begin{bmatrix}1\\-2\\1\end{bmatrix}, \quad [x]_B = \begin{bmatrix}0\\1\\0\end{bmatrix}$$

- S is l d. because $T = \left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}1\\-2\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix} \right\}$ is l. d in $\mathbb{R}^3$.
- Find generators for the relations in S $\longleftrightarrow$ find generators for the relations for T

$$\begin{bmatrix}1&1&1&0\\0&2&-2&1\\0&1&1&0\end{bmatrix} \xrightarrow[R_2 \leftrightarrow R_3]{} \begin{bmatrix}1&1&1&0\\0&1&1&0\\0&2&-2&1\end{bmatrix} \xrightarrow[R_3 \to R_3 - 2R_2]{} \begin{bmatrix}1&1&1&0\\0&1&1&0\\0&0&-4&1\end{bmatrix} \xrightarrow[R_3 \to -\frac{R_3}{4}]{} \begin{bmatrix}1&1&1&0\\0&1&1&0\\0&0&1&-\frac14\end{bmatrix}$$

$$\xrightarrow[\substack{R_2 \to R_2 - R_3 \\ R_1 \to R_1 - R_3}]{} \begin{bmatrix}1&1&0&\frac14\\0&1&0&\frac14\\0&0&1&-\frac14\end{bmatrix} \xrightarrow[R_1 \to R_1 - R_2]{} \begin{bmatrix}1&0&0&0\\0&1&0&\frac14\\0&0&1&-\frac14\end{bmatrix} \qquad \begin{aligned} x_1 &= 0 \\ x_2 &= -\tfrac14 x_4 \\ x_3 &= \tfrac14 x_4 \end{aligned}$$

All relations: $x_4 \left(-\frac14 \begin{bmatrix}1\\2\\1\end{bmatrix} + \frac14 \begin{bmatrix}1\\-2\\1\end{bmatrix} + \begin{bmatrix}0\\1\\0\end{bmatrix} \right) = \mathbb{0}$ for any x_4.

Consequence:
- $\left\{ \begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}1\\-2\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix} \right\}$ is l d. Any subset of 2 of these 3 vectors is l i
- $\left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}1\\-2\\1\end{bmatrix} \right\}$, $\left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix} \right\}$, $\left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\-2\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix} \right\}$ are l.i. $\underset{\dim \mathbb{R}^3 = 3}{\leadsto}$ they are bases for $\mathbb{R}^3$

Translation to V:
- $-\frac14 v_2 + \frac14 v_3 + v_4 = \mathbb{0}$ in $\mathcal{P}_2$
- $\{v_2, v_3, v_4\}$ is l d, but $\{v_2, v_3\}$, $\{v_2, v_4\}$, $\{v_3, v_4\}$ are l.i.
- $\{v_1, v_2, v_3\}$, $\{v_1, v_2, v_4\}$, $\{v_1, v_3, v_4\}$ are li in $\mathcal{P}_2$ $\underset{\dim \mathcal{P}_2 = 3}{\leadsto}$ they are bases for $\mathcal{P}_2$.

Want to write $[x^2]_{\{v_1, v_2, v_3\}} \longleftrightarrow$ Write $\begin{bmatrix}0\\0\\1\end{bmatrix} = [x^2]_B$ in $\left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}1\\-2\\1\end{bmatrix} \right\}$

$$\begin{bmatrix}0\\0\\1\end{bmatrix} = a \begin{bmatrix}1\\0\\0\end{bmatrix} + b \begin{bmatrix}1\\2\\1\end{bmatrix} + c \begin{bmatrix}1\\-2\\1\end{bmatrix} \qquad \left.\begin{aligned} 0 &= a+b+c \\ 0 &= 2(b-c) \\ 1 &= b+c \end{aligned}\right\} \; b = c = \tfrac12 \leadsto a = -1.$$

Translation: $x^2 = -v_1 + \frac12 v_2 + \frac12 v_3$.

Can check this: $x^2 \overset{?}{=} -1 + \frac12 (x+1)^2 + \frac12 (x-1)^2 = -1 + \frac12 (x^2+2x+1) + \frac12 (x^2-2x+1)$ ✓