Lecture 23: §3.7 Linear Transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$

GOAL: Study functions between (subspaces of) $\mathbb{R}^n$ & $\mathbb{R}^m$ that respect the vector space structure on both sides (that is, addition an scalar multiplication)

NOTE: Choice of coordinates for finite dimensional vector spaces V & W will allow us to extend the notion of linear maps to $F: \mathbb{V} \longrightarrow \mathbb{W}$ $\dim \mathbb{V} = n$, $\dim \mathbb{W} = m$

coords w.r.t fixed bases $B_{\mathbb{V}}$ & $B_{\mathbb{W}}$ $\longrightarrow$ $\mathbb{R}^n \longrightarrow \mathbb{R}^m$ linear

(We will see this in §5.7, next time)

We start with examples: everything will boil down to one prototypical example, namely multiplication by a fixed $m \times n$ matrix

§1 Examples:

Ex 1 $F: \mathbb{R}^3 \longrightarrow \mathbb{R}$ $(n=3, m=1)$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \longmapsto x_1 + 5x_3 = x_1 + 0\cdot x_2 + 5\cdot x_3$$

(this is a linear expression in the unknowns x_1, x_2, x_3)

We can realize it as multiplication by $A = [1\ 0\ 5]$ (1×3 matrix)

$$F(\underline{x}) = A\cdot \underline{x} = [1\ 0\ 5]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Q What does "F is linear" mean?

(1) $$F\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}\right) = F\left(\begin{bmatrix} x_1+y_1 \\ x_2+y_2 \\ x_3+y_3 \end{bmatrix}\right) = (x_1+y_1) + 5(x_3+y_3).$$
$$= (x_1+5x_3) + (y_1+5y_3) = F\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) + F\left(\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}\right)$$

So, the image of a sum of 2 vectors equals the sum of the image of these 2 vectors

(2) $$F\left(\alpha\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = F\left(\begin{bmatrix} \alpha x_1 \\ \alpha x_2 \\ \alpha x_3 \end{bmatrix}\right) = (\alpha x_1) + 5(\alpha x_3) = \alpha(x_1 + 5x_3) = \alpha F\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right)$$

So the image of a scaled vector is obtained by scaling the image of the vector.

Observation: We can restrict F to a line or a plane through the origin in $\mathbb{R}^3$, and get 2 "new" functions $F_1 = F|_L : \overset{\text{line}}{L} \longrightarrow \mathbb{R}$, $F_2 = F|_\Pi : \overset{\text{plane}}{\Pi} \longrightarrow \mathbb{R}$ (restriction to subspaces of $\mathbb{R}^3$ = domain of F)

How? • L line is the linear span of 1 nonzero vector $\vec{v}$ (direction of L) $v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$

So $F\left(\underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\text{in } L}\right) = F(\alpha \vec{v}) = \alpha F(\vec{v}) = \alpha (v_1 + 5 v_3)$.

For example: $v = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, so L is the x-axis in $\mathbb{R}^3$. Then, $F(L)$ is the whole $\mathbb{R}$ since $F\left(\begin{bmatrix} \alpha \\ 0 \\ 0 \end{bmatrix}\right) = \alpha$ & α is any real number

$v = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ Again, $F\left(\begin{bmatrix} \alpha \\ 2\alpha \\ 0 \end{bmatrix}\right) = \alpha$ so $F(L) = \mathbb{R}$

$v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ Now $F\left(\begin{bmatrix} \alpha \\ 2\alpha \\ 3\alpha \end{bmatrix}\right) = \alpha + 15\alpha = 16\alpha$ Again, $F(L) = \mathbb{R}$

• Π plane is the linear span of 2 l.i vectors in $\mathbb{R}^3$, $v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$, $u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$

So $F\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = F\left(\alpha \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} + \beta \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}\right) \underset{F \text{ linear}}{=} F\left(\alpha \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}\right) + F\left(\beta \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}\right)$

$\underset{F \text{ linear}}{=} \alpha F\left(\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}\right) + \beta F\left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}\right)$

these 2 vectors completely determine $F(\Pi)$

For example (a) $v = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $u = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $\quad F\left(\underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\text{in } \Pi}\right) = F\left(\begin{bmatrix} \alpha \\ \beta \\ 0 \end{bmatrix}\right) = \alpha = \alpha \underbrace{F\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)}_{=1}$

Π = xy-plane in $\mathbb{R}^3$

NOTE: $F(u) = 0$

Q What other vectors go to 0? Only need $\alpha = 0$, so only vectors mapping to 0 in $\mathbb{R}$ are those on the y-axis (that is, scalar multiples of $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$)

(b) $v = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $u = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ $\quad F\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = F\left(\begin{bmatrix} \alpha \\ 0 \\ \beta \end{bmatrix}\right) = \alpha + 5\beta$

Π = xz-plane in $\mathbb{R}^3$ $\quad = \alpha \underbrace{F\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)}_{=1} + \beta \underbrace{F\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)}_{=5}$

Q: What vectors in xz-plane map to 0? Need $\alpha + 5\beta = 0$ so $\alpha = -5\beta$

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = -5\beta \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \beta \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix}$

A The line L with direction $\begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix}$ through the origin maps to $\underline{0}$ in $\mathbb{R}$

Nothing else maps to 0.

In both cases: $F(\Pi) = \mathbb{R}$ & $\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{in } \mathbb{R}^3 : F\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = 0 \right\}$ = a line through $(0,0,0)$ in $\mathbb{R}^3$.

$\boxed{\text{Ex 2}}$ We can get a linear function $G : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ by 2 linear functions $F_1 : \mathbb{R}^3 \longrightarrow \mathbb{R}$ (1st coordinate for G). $\rightsquigarrow$ $G(\underline{x}) = \begin{bmatrix} F_1(\underline{x}) \\ F_2(\underline{x}) \end{bmatrix}$

$F_2 : \mathbb{R}^3 \longrightarrow \mathbb{R}$ (2nd ————).

<u>Example</u>: $F_1(\underline{x}) = x_1 + 5x_3 = \begin{bmatrix} 1 & 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$F_2(x) = 3x_1 - 7x_2 + 8x_3 = \begin{bmatrix} 3 & -7 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

Then $G\left(\underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\text{in } \mathbb{R}^3}\right) = \underbrace{\begin{bmatrix} x_1 + 5x_3 \\ 3x_1 - 7x_2 + 8x_3 \end{bmatrix}}_{\text{in } \mathbb{R}^2} = \underbrace{\begin{bmatrix} 1 & 0 & 5 \\ 3 & -7 & 8 \end{bmatrix}}_{=: A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

<u>Conclusion</u>: $G : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ becomes matrix multiplication by a <u>fixed</u> 2×3 matrix A (<u>NOTE</u>: # rows of $A = 2$ = dim target space $(= \mathbb{R}^2)$)

columns of $A = 3 =$ dim domain $(= \mathbb{R}^3)$

<u>Crucial observation</u>: What are the images of the canonical basis elements of $\mathbb{R}^3$? (e_1, e_2, e_3)

- $G\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ = 1st column of A
- $G\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -7 \end{bmatrix}$ = 2nd column of A
- $G\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$ = 3rd column of A

<u>In other words</u> The 3 vectors $G(e_1), G(e_2)$ & $G(e_3)$ <u>determine the matrix A & hence the map G</u>

<u>Q</u>: Another way of seeing this?

$$G\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = G\Big(\underbrace{x\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{\text{sum of 3 vectors}}\Big) \underset{G \text{ linear}}{=} G\left(x\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) + G\left(y\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) + G\left(z\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)$$

$$\underset{G \text{ linear}}{=} x\underbrace{\boxed{G\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)}}_{col_1(A)} + y\underbrace{\boxed{G\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right)}}_{col_2(A)} + z\underbrace{\boxed{G\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)}}_{col_3(A)}$$

$$= A \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Conclude <u>The 3 boxed vectors determine $G\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)$</u> because they determine the matrix A we were looking for earlier.

Q: What vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ go to $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ under G?

To answer, need to solve $A\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightsquigarrow$ A: Null Space of A!!!

In this example: $\begin{bmatrix} 1 & 0 & 5 \\ 3 & -7 & 8 \end{bmatrix} \xrightarrow[R_2 \to R_2 - 3R_1]{} \begin{bmatrix} 1 & 0 & 5 \\ 0 & -7 & -7 \end{bmatrix} \xrightarrow[R_2 \to \frac{R_2}{-7}]{} \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 1 \end{bmatrix}$ indep $\quad x_1 = -5x_3$, $x_2 = -x_3$

So $\mathrm{Null}(A) = \left\{ x_3 \begin{bmatrix} -5 \\ -1 \\ 1 \end{bmatrix} : x_3 \text{ arbitrary} \right\} = \left\langle \begin{bmatrix} -5 \\ -1 \\ 1 \end{bmatrix} \right\rangle$ (Check: $G\begin{pmatrix} -5 \\ -1 \\ 1 \end{pmatrix} = \begin{bmatrix} -5+5 \\ -15+7+8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ ✓)

Q: What vectors lie in the image of G in $\mathbb{R}^2$?

To answer this, we have to find vectors of the form $G\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right)$ as we vary $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ in $\mathbb{R}^3$.

$G\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = A\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 5 \\ 3 & -7 & 8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1\,\mathrm{col}_1(A) + x_2\,\mathrm{col}_2(A) + x_3\,\mathrm{col}_3(A)$ for x_1, x_2, x_3 in $\mathbb{R}$

So Image of G = Column Space of A !!!

NOTE: $\mathrm{rank}(A) = 2$ in $\mathrm{ColSp}(A)$ lies in $\mathbb{R}^2$, so Image $G = \mathbb{R}^2$
$\quad \parallel$
$\dim \mathrm{ColSp}(A)$

In both examples:
- Maps are determine by multiplication by a fixed matrix A
- Image of map = Column Space of A
- vectors mapping to $\underline{0}$ = Null Space of A

These 3 things will be true for any linear map $T: \mathbb{R}^m \longrightarrow \mathbb{R}^n$.

§2. General definition:

Def: $T: \mathbb{R}^m \longrightarrow \mathbb{R}^n$ function is a linear tranformation if 2 conditions hold

(1) $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$ for all $\vec{u}, \vec{v}$ in $\mathbb{R}^m$ (addition is respected)

(2) $T(\alpha\vec{u}) = \alpha T(\vec{u})$ for all $\vec{u}$ in $\mathbb{R}^m$ & α in $\mathbb{R}$ (scalar mult. is respected)

Non-example 1: $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ $\quad F(\underline{x}) = \begin{bmatrix} x_1 - x_2 + 1 \\ x_2 \\ 2x_1 + x_2 \end{bmatrix}$ is NOT linear $\quad \to$ problem!

$F\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$, $F\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ and $F\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = F\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$

BUT $F\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + F\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \neq \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} = F\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$ $\rightsquigarrow$ Condition (1) fails

Non-example 2: $F:\mathbb{R}\longrightarrow\mathbb{R}$ $F(x)=e^x$ is non-linear

$F(0)=1$, $F(1)=e$ $F(0)+F(1)=1+e\neq e=F(0+1)$

$F(2)=e^2\neq 2e=2F(1)$

(1) & (2) fail.

§3 Special cases: $T:\mathbb{R}^m\longrightarrow\mathbb{R}^n$ linear

- $\boxed{n=m=1}$ $\rightsquigarrow$ $T:\mathbb{R}\longrightarrow\mathbb{R}$ linear Q: What does the formula for T look like?

Prop 1: $T:\mathbb{R}\longrightarrow\mathbb{R}$ is linear if and only if $T(x)=ax$ for a fixed a in $\mathbb{R}$. Moreover, $a=T(1)$.

Proof: • $T(x)=ax$ is clearly linear. $T(x+y)=a(x+y)=ax+ay=T(x)+T(y)$

. $T(\alpha x)=a\alpha x=\alpha ax=\alpha T(x)$

• If T is linear, then $T(x)=T(x\cdot 1)=x\underbrace{T(1)}_{=:a}=ax$ for $a=T(1)$.

scalar!

- $\boxed{m=1,\ n\text{ arbitrary}}$ $\rightsquigarrow$ $T:\mathbb{R}^n\longrightarrow\mathbb{R}$ linear Q: What does it look like?

Prop 2: $T:\mathbb{R}^n\longrightarrow\mathbb{R}$ linear if and only if $T(x)=\underbrace{\vec{u}^T}_{=1\times n\text{ matrix}}x$ for some vector $\vec{u}$ in $\mathbb{R}^n$. Moreover $\vec{u}=\begin{bmatrix}T(e_1)\\T(e_2)\\\vdots\\T(e_n)\end{bmatrix}$

Proof: • $x\longmapsto\boxed{\vec{u}^T}x$ is a linear map

FIXED

• Now, assume we have a map $T:\mathbb{R}^n\longrightarrow\mathbb{R}$ that is linear.

Then: $T\left(\begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix}\right)=T\Bigg(\underbrace{x_1\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}+x_2\begin{bmatrix}0\\1\\0\\\vdots\\0\end{bmatrix}+\dots+x_n\begin{bmatrix}0\\\vdots\\0\\1\end{bmatrix}}_{\text{sum of } n \text{ vectors in } \mathbb{R}^n}\Bigg)$

BUT, T is linear so

$$T\left(x_1\begin{bmatrix}1\\0\\ \vdots\\0\end{bmatrix}+\cdots+x_n\begin{bmatrix}0\\ \vdots\\0\\1\end{bmatrix}\right)=T\left(x_1\begin{bmatrix}1\\0\\ \vdots\\0\end{bmatrix}\right)+T\left(x_2\begin{bmatrix}0\\1\\0\\ \vdots\\0\end{bmatrix}\right)+\cdots+T\left(x_n\begin{bmatrix}0\\0\\ \vdots\\0\\1\end{bmatrix}\right)$$

$$\underset{T \text{ linear}}{=} x_1 \underbrace{T\left(\begin{bmatrix}1\\0\\ \vdots\\0\end{bmatrix}\right)}_{=:u_1} + x_2 \underbrace{T\left(\begin{bmatrix}0\\1\\0\\ \vdots\\0\end{bmatrix}\right)}_{=:u_2} + \cdots + x_n \underbrace{T\left(\begin{bmatrix}0\\0\\ \vdots\\0\\1\end{bmatrix}\right)}_{=:u_n}$$

$$= u_1x_1 + u_2x_2 + \cdots + u_nx_n = \vec{u}^T\begin{bmatrix}x_1\\ \vdots\\x_n\end{bmatrix}.$$

§3 General form of $T:\mathbb{R}^n \longrightarrow \mathbb{R}^m$ linear

- We build T from m linear functions
$T_1 : \mathbb{R}^n \longrightarrow \mathbb{R}$ (1st coord)
$T_2 : \mathbb{R}^n \longrightarrow \mathbb{R}$ (2nd coord)
$\vdots$
$T_m : \mathbb{R}^n \longrightarrow \mathbb{R}$ (last coord)

By Prop 2:

$$T_1(x) = \vec{v}_1^{\,T}\begin{bmatrix}x_1\\ \vdots\\x_n\end{bmatrix} \qquad \vec{v}_1 \text{ in } \mathbb{R}^n$$

$$T_2(x) = \vec{v}_2^{\,T}\begin{bmatrix}x_1\\ \vdots\\x_n\end{bmatrix} \qquad \vec{v}_2 \text{ in } \mathbb{R}^n$$

$$\vdots$$

$$T_m(x) = \vec{v}_m^{\,T}\begin{bmatrix}x_1\\ \vdots\\x_n\end{bmatrix} \qquad \vec{v}_m \text{ in } \mathbb{R}^n$$

So
$$T(\underline{x}) = \begin{bmatrix}T_1(x)\\T_2(x)\\ \vdots\\T_m(x)\end{bmatrix} = \begin{bmatrix}\vec{v}_1^{\,T}\cdot\underline{x}\\ \vec{v}_2^{\,T}\cdot\underline{x}\\ \vdots\\ \vec{v}_m^{\,T}\cdot\underline{x}\end{bmatrix} = \underbrace{\begin{bmatrix}\vec{v}_1^{\,T}\\ \vdots\\ \vec{v}_m^{\,T}\end{bmatrix}}_{=A\ m\times n \text{ matrix!}}\begin{bmatrix}x_1\\ \vdots\\x_n\end{bmatrix}$$

Theorem: Every linear transformation $T:\mathbb{R}^n \longrightarrow \mathbb{R}^m$ has the form $T(\underline{x}) = A\begin{bmatrix}x_1\\ \vdots\\x_n\end{bmatrix}$ for some $m\times n$ matrix A.

Moreover: $\operatorname{col}_1(A) = T\left(\begin{bmatrix}1\\0\\ \vdots\\0\end{bmatrix}\right)$, $\operatorname{col}_2(A) = T\left(\begin{bmatrix}0\\1\\0\\ \vdots\\0\end{bmatrix}\right)$, $\ldots$, $\operatorname{col}_n(A) = T\left(\begin{bmatrix}0\\ \vdots\\0\\1\end{bmatrix}\right)$

so $A = [\,T(e_1), \ldots, T(e_n)\,]$ (Columns are the image of the canonical basis!)

Proof: - First part we know from the earlier discussion

- For the second part, we only need to use that $\{e_1, \dots, e_n\}$ is a basis for $\mathbb{R}^n$ & that T is linear

$$T\left(\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}\right) \overset{\text{basis property}}{=} T\left(x_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \dots + x_n \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}\right)$$

$$= T\left(x_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right) + \dots + T\left(x_n \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}\right)$$

$$= x_1 \boxed{T(e_1)} + \dots + x_n \boxed{T(e_n)}$$

$$= \boxed{\begin{bmatrix} T(e_1) & \cdots & T(e_n) \end{bmatrix}} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

This has to be the matrix A.

Exercise: Find a linear transformation $F: \mathbb{R}^2 \to \mathbb{R}^3$ such that $F\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$, $F\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix}$

Is F unique?

Solution: Write e_1 & e_2 in terms of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ & $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

(We can do this because $\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$ is a basis for $\mathbb{R}^2$)

$$\begin{cases} e_1 = 2\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ e_2 = -\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} \end{cases}$$

so $F(e_1) = 2F\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) - F\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$

$$= 2\begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \\ 1 \end{bmatrix}$$

so $F(e_2) = -F\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) + F\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$

$$= -\begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ -4 \\ 2 \end{bmatrix}$$

Conclude by the Theorem that $\boxed{F\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 8 & -4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}$

In particular, it is unique!

§4 Null Space & Range

Fix $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ linear transformation (so we can view $T(\underline{x}) = A\underline{x}$ for some $m \times n$ matrix A)

We can associate two natural subspaces to T = Null Space & Range

Def 1: The Null Space (or Kernel) of T is the set of vectors $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ in $\mathbb{R}^n$ mapping to $\underline{0} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ in $\mathbb{R}^m$ under T, that is

$$\mathcal{N}(T) = \{ \vec{v} \text{ in } \mathbb{R}^n : T(\vec{v}) = \vec{0} \text{ in } \mathbb{R}^n \}$$

Def 2: The Range of T is the image of T, that is

$$R(T) = \{ \vec{w} \text{ in } \mathbb{R}^m : \text{we can find } \vec{u} \text{ in } \mathbb{R}^n \text{ with } T(\vec{u}) = \vec{w} \}$$

Our interpretation of T as multiplication by A yields:

Theorem 2: • $\mathcal{N}(T) = \mathcal{N}(A)$ (nullspace of A) & $R(T) = ColSp(A)$ (= Range of A)

In particular, • $\mathcal{N}(T)$ is a subspace of $\mathbb{R}^n$ with $\dim = null(A)$

• $R(T)$ ——— $\mathbb{R}^m$ ——— $= rank(A)$

We define $nullity(T) = \dim \mathcal{N}(T)$ (= $nullity(A)$)

$rank(T) = \dim R(T)$ (= $rank(A)$)

Corollary: $\dim \mathcal{N}(T) + \dim R(T) = n$ (= # cols(A))