Lecture XIV: §3.7 (cont) Matrix representations of linear maps
§3.8 Examples for linear maps between abstract vector spaces

§1. Matrix Representations:

Recall: A linear transformation $T:\mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a map satisfying:
$$\vec{v} \longmapsto T(\vec{v})$$

(1) $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$ for all $\vec{u},\vec{v}$ in $\mathbb{R}^n$

(2) $T(\alpha\vec{u}) = \alpha\, T(\vec{u})$ for all $\vec{u}$ in $\mathbb{R}^n$, $\alpha\in\mathbb{R}$.

Obs: $\boxed{T(\underbrace{\vec{0}}_{\text{in } \mathbb{R}^n}) = \vec{0} \text{ in } \mathbb{R}^m}$ because $\vec{0} = 0\cdot\vec{0}$ in both $\mathbb{R}^n$ & $\mathbb{R}^m$

& $\vec{0} = \underset{\text{scalar}}{0}\,\vec{w}$ for any $\vec{w}$ in $\mathbb{R}^m$

so $T(\vec{0}) = T(0\cdot\vec{0}) \overset{(2)}{=} 0\,\overbrace{T(\vec{0})}^{\vec{w} \text{ in } \mathbb{R}^m} = \vec{0}$ in $\mathbb{R}^m$.

Consequence: $\vec{0}$ in $\mathcal{N}(T)$ & $\vec{0}$ in $\mathcal{R}(T)$ (We know this because $\mathcal{N}(T)$ & $\mathcal{R}(T)$ are subspaces!)

Matrix Representation Theorem: Any linear transformation $T:\mathbb{R}^n \longrightarrow \mathbb{R}^m$ can be written as $T(\vec{v}) = A\cdot\vec{v}$, where the matrix A of size $m\times n$ has the form $A = [\,T(e_1)\ \cdots\ T(e_n)\,]$

canonical basis elements in $\mathbb{R}^n$ (in the given order!)

Alternatively: A = coefficient matrix for the linear expression in each coord. of $T\left(\begin{bmatrix} x_1 \\ \vdots \\ x_n\end{bmatrix}\right)$.

Example: $T:\mathbb{R}^3 \longrightarrow \mathbb{R}^2$ $\quad T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3\end{bmatrix}\right) = \begin{bmatrix} x_1+2x_3 \\ 2x_2-5x_3\end{bmatrix} \rightsquigarrow A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & -5\end{bmatrix}$

Here, $T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}1\\0\end{bmatrix}$, $T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}0\\2\end{bmatrix}$ & $T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}2\\-5\end{bmatrix}$

Conclusion: T is completely determined by its values on $\{e_1,\ldots,e_n\}$

But, why is the canonical basis better than any other basis for $\mathbb{R}^n$?

Answer: It is not!

Proposition: Given a basis $B = \{\vec{v}_1,\ldots,\vec{v}_n\}$ for $\mathbb{R}^n$ & ANY $\{\vec{w}_1,\ldots,\vec{w}_n\}$ in $\mathbb{R}^m$, there is a unique linear transformation $T:\mathbb{R}^n\longrightarrow\mathbb{R}^m$ with $T(\vec{v}_1)=\vec{w}_1,\ldots,T(\vec{v}_n)=\vec{w}_n$

same number of vectors as $\dim \mathbb{R}^n = n$

Example (last time): $T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \begin{bmatrix}1\\4\\3\end{bmatrix}$, $T\left(\begin{bmatrix}1\\2\end{bmatrix}\right) = \begin{bmatrix}1\\0\\5\end{bmatrix}$ determine a unique linear transf $T:\mathbb{R}^2\longrightarrow\mathbb{R}^3$ because $\left\{\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}1\\2\end{bmatrix}\right\}$ is a basis for $\mathbb{R}^2$. $\underline{A}$: $T(\vec{v}) = \begin{bmatrix}1 & 0\\ 8 & -4\\ 1 & 2\end{bmatrix}\vec{v}$

⚠ This is not true if $\{v_1,\dots,v_n\}$ is not a basis.

Ex $\vec{v}_1=\begin{bmatrix}1\\0\end{bmatrix}$ $\vec{v}_2=\begin{bmatrix}2\\0\end{bmatrix}$ l.d vectors in $\mathbb{R}^2$

$\vec{w}_1=\begin{bmatrix}1\\0\\0\end{bmatrix}$ $\vec{w}_2=\begin{bmatrix}1\\0\\0\end{bmatrix}$

We cannot have a linear transf $T:\mathbb{R}^2\to\mathbb{R}^3$ with $T(\vec{v}_1)=\vec{w}_1$ & $T(\vec{v}_2)=\vec{w}_2$

Why? $\vec{w}_2=T(\vec{v}_2)=T(2\cdot\underbrace{\begin{bmatrix}1\\0\end{bmatrix}}_{=v_1}) \underset{T\text{ linear}}{=} 2T(\underbrace{\begin{bmatrix}1\\0\end{bmatrix}}_{\vec{v}_1})=2\cdot\vec{w}_1$

but $\vec{w}_2\neq 2\vec{w}_1$

The problem: explicit linear dependency for $\vec{v}_1,\vec{v}_2$ does NOT hold for $\vec{w}_1,\vec{w}_2$

Another example: We cannot have $T:\mathbb{R}^2\to\mathbb{R}^3$ linear with $T(\begin{bmatrix}1\\0\end{bmatrix})=\begin{bmatrix}2\\0\\0\end{bmatrix}$, $T(\begin{bmatrix}0\\1\end{bmatrix})=\begin{bmatrix}1\\0\\1\end{bmatrix}$ & $T(\begin{bmatrix}1\\1\end{bmatrix})=\begin{bmatrix}3\\1\\2\end{bmatrix}$.

Why? $\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}+\begin{bmatrix}0\\1\end{bmatrix}$ $\underset{\text{apply }T}{\leadsto}$ $T(\begin{bmatrix}1\\1\end{bmatrix})=T(\begin{bmatrix}1\\0\end{bmatrix})+T(\begin{bmatrix}0\\1\end{bmatrix})$

$\begin{bmatrix}3\\1\\2\end{bmatrix} = \begin{bmatrix}2\\0\\0\end{bmatrix}+\begin{bmatrix}1\\0\\1\end{bmatrix}$ This is false!

So T linear with the prescribed assigned values for e_1, e_2 & $\begin{bmatrix}1\\1\end{bmatrix}$ cannot exist.

Proof of Prop: Since $B=\{\vec{v}_1,\dots,\vec{v}_n\}$ is a basis for $\mathbb{R}^n$, any vector $\vec{v}$ in $\mathbb{R}^n$ can be uniquely written as $\boxed{\vec{v}=\alpha_1\vec{v}_1+\dots+\alpha_n\vec{v}_n}$ for suitable $\alpha_1,\dots,\alpha_n$.

(Recall: $[v]_B=\begin{bmatrix}\alpha_1\\\vdots\\\alpha_n\end{bmatrix}$ coordinates of $\vec{v}$ with respect to the basis B)

Then, "apply T" to the boxed expression & use the linearity properties

(Remember, we are trying to guess the value of $T(\vec{v})$!!)

$$T(\vec{v})=T(\alpha_1\vec{v}_1+\dots+\alpha_n\vec{v}_n)\underset{\text{Prop (1)}}{=}T(\alpha_1\vec{v}_1)+\dots+T(\alpha_n\vec{v}_n)$$

$$\underset{\text{Prop (2)}}{=}\alpha_1\underbrace{T(\vec{v}_1)}_{=\vec{w}_1}+\dots+\alpha_n\underbrace{T(\vec{v}_n)}_{=\vec{w}_n}$$

(these are the prescribed values for T!)

Conclude: $T(\vec{v})=\underbrace{[\vec{w}_1\ \dots\ \vec{w}_n]}_{\text{matrix of }T(\vec{v}_1),\dots,T(\vec{v}_n)}[\vec{v}]_B$

Can check: This map T is linear $\left(\begin{array}{l}T(\vec{v}+\vec{u})=T(\vec{v})+T(\vec{u})\\ T(\beta\vec{v})=\beta T(\vec{v})\end{array}\right)$

The reason: Taking coordinates with respect to a basis is linear!

Given $\mathcal{B}$ basis for $\mathbb{R}^n$, $F: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is linear.
$\vec{v} \longmapsto [\vec{v}]_{\mathcal{B}}$

Q: Why are matrix representations useful?
A: Allow for fast compositions!

same number!!

Prop: Given $F: \mathbb{R}^n \longrightarrow \mathbb{R}^{\boxed{m}}$, $G: \mathbb{R}^{\boxed{m}} \longrightarrow \mathbb{R}^s$ two linear transf

then the composition $G \circ F: \mathbb{R}^n \longrightarrow \mathbb{R}^s$ defined by

$\left(G\circ F: \mathbb{R}^n \xrightarrow{F} \mathbb{R}^m \xrightarrow{G} \mathbb{R}^s \right.$ apply F first & then G $\left.\right)$

$\underset{\text{in } \mathbb{R}^n}{\vec{v}} \longmapsto G(\underbrace{F(\vec{v})}_{\text{in } \mathbb{R}^m})$ in $\mathbb{R}^s$

is also a linear transformation.

Furthermore, if $F(\vec{v}) = A\vec{v}$ A of size $m\times n$, then
$G(\vec{w}) = B\vec{w}$ B —— $s\times m$

the matrix representing $G\circ F$ is BA (size $s\times n$).

$\boxed{\text{VERY IMPORTANT:}}$ We must multiply B & A in the same order as we compose G & F.

Before we discuss the proof, let's look at an example.

Example: $F: \mathbb{R}^4 \longrightarrow \mathbb{R}^2$ $\quad F\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} x_1 - x_2 \\ x_3 + x_4 \end{bmatrix}$ $\quad A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

$G: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ $\quad G\left(\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right) = \begin{bmatrix} y_1 \\ y_1 + 5y_2 \\ -y_1 + y_2 \end{bmatrix}$ $\quad B = \begin{bmatrix} 1 & 0 \\ 1 & 5 \\ -1 & 1 \end{bmatrix}$

So $BA = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & -1 & 5 & 5 \\ -1 & 1 & 1 & 1 \end{bmatrix}$

replace $y_1 = (x_1 - x_2)$, $y_2 = (x_3 + x_4)$ in formula for G

And $G\circ F: \mathbb{R}^4 \longrightarrow \mathbb{R}^3$ is $G\circ F\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = G\left(\begin{bmatrix} x_1 - x_2 \\ x_3 + x_4 \end{bmatrix}\right) = \begin{bmatrix} (x_1 - x_2) \\ x_1 - x_2 + 5(x_3 + x_4) \\ -(x_1 + x_2) + (x_3 + x_4) \end{bmatrix}$

$= \begin{bmatrix} x_1 - x_2 \\ x_1 - x_2 + 5x_3 + 5x_4 \\ -x_1 + x_2 + x_3 + x_4 \end{bmatrix}$

Check: matrix for $G\circ F$ must have size 3×4 & $= \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & -1 & 5 & 5 \\ -1 & 1 & 1 & 1 \end{bmatrix}$,
just as the Prop predicted!

Proof: Check 2 linear properties

(1) $G(F(\vec{v}+\vec{u})) \underset{F \text{ linear}}{=} G(\underbrace{F(\vec{v})}_{=\vec{w}_1} + \underbrace{F(\vec{u})}_{\vec{w}_2}) \underset{G \text{ linear}}{=} G(F(\vec{v})) + G(F(\vec{u}))$ ✓

(2) $G(F(\alpha\vec{v})) \underset{F \text{ linear}}{=} G(\alpha \underbrace{F(\vec{v})}_{\vec{w} \text{ in } \mathbb{R}^m}) \underset{G \text{ linear}}{=} \alpha\, G(F(\vec{v}))$ ✓

So we know $G\circ F$ has a matrix representing it. Indeed, the matrix is $[\,G\circ F(e_1) \;\cdots\; G\circ F(e_n)\,]$

We compute each column vector:

$$G\circ F(e_1) \underset{\text{def of } F}{=} G\left(A\begin{bmatrix}1\\0\\ \vdots\\0\end{bmatrix}\right) = G(\underbrace{\text{col}_1(A)}_{\vec{w}_1 \text{ in } \mathbb{R}^m}) \underset{\text{def of } G}{=} B\,\text{col}_1(A)$$

$$\vdots$$

$$G\circ F(e_n) = G\left(A\begin{bmatrix}0\\ \vdots\\0\\1\end{bmatrix}\right) = G(\underbrace{\text{col}_n(A)}_{=\vec{w}_n \text{ in } \mathbb{R}^m}) \underset{\text{def of } G}{=} B\,\text{col}_n(A)$$

so $[\,G\circ F(e_1) \cdots G\circ F(e_n)\,] = [\,\underbrace{B\,\text{col}_1(A)}_{\text{col}_1(BA)} \cdots \underbrace{B\,\text{col}_n(A)}_{\text{col}_n(BA)}\,] \underset{\text{def of matrix multiplication}}{=} BA$ size $s\times n$

This size is consistent with $G\circ F: \mathbb{R}^n \longrightarrow \mathbb{R}^s$. □

§2. Linear transformation for abstract vector spaces:

We can easily generalize linear transformations $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ to linear transformations between abstract vector spaces because we have a way to add & scale multiply vectors in abstract vector spaces. Our maps $T: V \longrightarrow W$ will be linear if the respect these 2 operations. More precisely:

Def: Fix V, W two (abstract) vector spaces & let $T: V \longrightarrow W$, $\vec{v} \longmapsto T(\vec{v})$ be a map. We say T is a linear transformation if

(1) $T(\vec{v} +_V \vec{u}) = T(\vec{v}) +_W T(\vec{u})$ for all $\vec{u}, \vec{v}$ in V & (2) $T(\alpha \cdot_V \vec{v}) = \alpha \cdot_W T(\vec{v})$ for $\vec{v}$ in V, α in $\mathbb{R}$

($+_V$: sum in V; $+_W$: sum in W; $\cdot_V$: scalar mult in V; $\cdot_W$: in W)

For the rest of today's lecture, we'll focus on examples.

Example 0 Take $\mathbb{V} = \mathbb{R}^n$, $\mathbb{W} = \mathbb{R}^m$. Then $T: \mathbb{R}^n \to \mathbb{R}^m$ linear in the usual sense, so it's just multiplication by an $m \times n$ matrix.

Example 1 $T: \mathcal{P}_2 \longrightarrow \mathbb{R}$ $\quad T(P_{(x)}) = P_{(1)}$

↳ polynomials of deg ≤ 2

Claim: T is linear

Why? $T(P_{(x)} + Q_{(x)}) = (P+Q)_{(1)} = P_{(1)} + Q_{(1)} = T(P_{(x)}) + T(Q_{(x)})$

(new poly; because of how we defined addition of polynomials)

$T(\underbrace{\alpha P_{(x)}}_{\text{new polynomial}}) = (\alpha P)_{(1)} = \alpha P_{(1)} = \alpha T(P_{(x)})$ ✓

(because of how we defined scalar mult in $\mathbb{R}^2$)

- These statements are true but they don't really help us understand what T is.

Better way: write $P_{(x)} = a + bx + cx^2 \rightsquigarrow P_{(1)} = a+b+c$

So we can really view T as the linear map $\tilde{T}: \mathbb{R}^3 \longrightarrow \mathbb{R}$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \longmapsto a+b+c$$

Here $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = [P]_B$ coordinates of P in the basis $\{1, x, x^2\}$ for $\mathcal{P}_2$

- This example generalizes to $T: C_{[-2,3]} \longrightarrow \mathbb{R}$

$$f \longmapsto f(1)$$

(we can replace 1 by any fixed number x_0 in $[-2,3]$)

We call it an evaluation map at $x_0 = 1$. It is linear!

Example 2: Taking coordinates with respect to a fixed basis B for $\mathbb{V}$ where $\dim \mathbb{V}$ is finite

Fix $\dim \mathbb{V} = p$. Then $T: \mathbb{V} \longrightarrow \mathbb{R}^p$ is a linear transf.

$$\vec{v} \longmapsto [\vec{v}]_B$$

Why? $\vec{v} = \alpha_1 \vec{v}_1 + \dots + \alpha_p \vec{v}_p \rightsquigarrow [\vec{v}]_B = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_p \end{bmatrix}$

$+ \quad \vec{w} = \beta_1 \vec{v}_1 + \dots + \beta_p \vec{v}_p \rightsquigarrow [\vec{w}]_B = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_p \end{bmatrix}$

$\vec{v} + \vec{w} = (\alpha_1 + \beta_1)\vec{v}_1 + \dots + (\alpha_p + \beta_p)\vec{v}_p \rightsquigarrow$ (1) $\boxed{[\vec{v} + \vec{w}]_B = \begin{bmatrix} \alpha_1 + \beta_1 \\ \vdots \\ \alpha_p + \beta_p \end{bmatrix}} = [\vec{v}]_B + [\vec{w}]_B$ ✓

(sum scalars columnwise)

Also $\alpha \vec{w} = (\alpha\beta_1)\vec{v}_1 + \dots + (\alpha\beta_p)\vec{v}_p \rightsquigarrow$ (2) $\boxed{[\alpha \vec{w}]_B = \begin{bmatrix} \alpha\beta_1 \\ \vdots \\ \alpha\beta_p \end{bmatrix} = \alpha [\vec{w}]_B}$ ✓

Obs: We can combine Examples 1 & 2 via composition

$$\begin{array}{ccc} T: \mathcal{P}_2 & \longrightarrow & \mathbb{R} \\ F \downarrow & & \| \\ \tilde{T}: \mathbb{R}^3 & \longrightarrow & \mathbb{R} \end{array}$$

$$F = \mathcal{P}_2 \longrightarrow \mathbb{R}^3$$
$$P \longmapsto \underset{=B}{[P]_{\{1,x,x^2\}}}$$

Take $B = \{1, x, x^2\}$ standard basis for $\mathcal{P}_2$.

$$T(a+bx+cx^2) = a+b+c$$
$$\tilde{T}\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = a+b+c$$

$\rightsquigarrow$ $\boxed{\text{Translation: For any } P_{(x)} \text{ in } \mathcal{P}_2 \text{ we have } T_{(P)} = \tilde{T}([P]_B) = \tilde{T} \circ F_{(P)}}$

KEY FACT
We will be able to do a similar translation from $T: V \longrightarrow W$ with $\dim V = n$, $\dim W = m$ to $\tilde{T}: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ by identifying V with $\mathbb{R}^n$ via coordinates wrt a basis B_V for V
W with $\mathbb{R}^m$ ——— B_W for W
(More on this, next time)

Example 3

$$T: C_{[0,1]} \longrightarrow \mathbb{R}$$
$$f_{(x)} \longmapsto \int_0^1 f_{(x)}\,dx$$

is a linear transformation

Example 4

$$T: C'_{(0,1)} \longrightarrow C_{(0,1)}$$
$$f_{(x)} \longmapsto f'_{(x)}$$

(Recall: $C'_{(0,1)}$ = continuous functions $f: (0,1) \to \mathbb{R}$ with $f': (0,1) \to \mathbb{R}$ also continuous)

T is a linear transformation

Example 5

$$T: M_{2\times 3} \longrightarrow M_{4\times 3}$$
$$A \longmapsto \boxed{\begin{bmatrix} 1 & 3 \\ 0 & 4 \\ 0 & 5 \\ 0 & 6 \end{bmatrix}} A$$

is a linear transform.

(Left multiplication by a fixed matrix of appropriate size)

$\hookrightarrow$ replacing this by any other fixed matrix of size 4×2 will have the same effect.

Example 6 $T_1 : \mathbb{V} \longrightarrow \mathbb{V}$ is linear & $T_2 : \mathbb{V} \longrightarrow \mathbb{W}$ is linear

$\vec{v} \longmapsto \vec{v}$ (identity map) $\qquad \vec{v} \longmapsto \mathbb{O}_{\mathbb{W}}$ (zero map)

§3 Basic Properties:

Theorem 1: Assume $\mathbb{V}$ has finite dimension, and basis $\mathcal{B} = \{v_1, \ldots, v_p\}$

Then, given any set $\{\vec{w}_1, \ldots, \vec{w}_p\}$ of vectors in $\mathbb{W}$, we can find a UNIQUE linear transformation $T : \mathbb{V} \longrightarrow \mathbb{W}$ with $T(\vec{v}_1) = \vec{w}_1$, $\ldots$, $T(\vec{v}_p) = \vec{w}_p$.

Obs: We don't need $\mathbb{W}$ to have finite dimension, only $\mathbb{V}$.

Why is Theorem 1 true? Use coordinates with respect to $\mathcal{B}$ & result when $\mathbb{V} = \mathbb{R}^n$, $\mathbb{W} = \mathbb{R}^m$

$$T(\vec{v}) = T(\alpha_1 \vec{v}_1 + \cdots + \alpha_p \vec{v}_p) \underset{T \text{ linear!}}{=} \alpha_1 \underbrace{\boxed{T(\vec{v}_1)}}_{= \vec{w}_1} + \cdots + \alpha_p \underbrace{\boxed{T(\vec{v}_p)}}_{= \vec{w}_p}$$

So $T(\vec{v}) = \alpha_1 \vec{w}_1 + \cdots + \alpha_p \vec{w}_p$ where $[v]_{\mathcal{B}} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_p \end{bmatrix}$

Check: The map T defined by this formula is linear

- Uniqueness follows because we found a unique possibility for defining $T(\vec{v})$ using $\mathcal{B}$ & $\vec{w}_1, \ldots, \vec{w}_p$.

Application: Find a linear transformation $T : \mathbb{P}_3 \longrightarrow \mathbb{P}_2$ with

$T(1) = 2 + x$, $T(x) = x - x^2$, $T(x^2) = 5 - 10x$ & $T(x^3) = 2$.

Solution: $T(a + bx + cx^2 + dx^3) \overset{T \text{ linear}}{=} a\,T(1) + b\,T(x) + c\,T(x^2) + d\,T(x^3)$

$= a(2+x) + b(x - x^2) + c(5 - 10x) + d \cdot 2$

we have preassigned values for $T(1), T(x), T(x^2)$ & $T(x^3)$

$= (2a + 5c + 2d) + (a - b - 10c)\,x + (-b)\,x^2$

Note: If we fix $B = \{1, x, x^2, x^3\}$ basis for V, we can
$B_2 = \{1, x, x^2\}$ ——— W

identify $T : \mathcal{P}_3 \longrightarrow \mathcal{P}_2$ with $\tilde{T} : \mathbb{R}^4 \longrightarrow \mathbb{R}^3$

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \longmapsto \begin{bmatrix} 2a+5c+2d \\ a+b-10c \\ -b \end{bmatrix}$$

Here $[a + bx + cx^2 + dx^3]_B = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$

$$[T(a+bx+cx^2+dx^3)]_{B_2} = \begin{bmatrix} 2a+5c+2d \\ a+b-10c \\ -b \end{bmatrix}$$

So we get a "commutative diagram" we going from $V \longrightarrow \mathbb{R}^3$ in 2 ways must be the same!

way 1

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "{[\,]_B}"', "\cong"] \arrow[dr, phantom, "\circlearrowleft"] & W \arrow[d, "{[\,]_{B_2}}", "\cong"'] \\ \mathbb{R}^4 \arrow[r, "\tilde{T}"'] & \mathbb{R}^3 \end{tikzcd}$$

way 2

$\rightsquigarrow$ $[T(p)]_{B_2} = \tilde{T}([p]_B)$

(way 1) (way 2)

Upshot: Studying linear transformations between vectors spaces of finite dimension is as EASY as studying linear maps $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$
Just pick coordinates on V & W. !!

Next time: • Null Spaces & Range for $T : V \longrightarrow W$ (rest of §5.7)
• Operations with linear transformation. (§5.8)