Lecture 25: §5.7 Null Space & Range of Linear transformations
Rank - Nullity Theorem

Recall Last time we defined linear transformations $T: \mathbb{V} \longrightarrow \mathbb{W}$ for abstract vector spaces: (1) $T(\vec{v}+\vec{w}) = T(\vec{v}) + T(\vec{w})$ for all $\vec{v}, \vec{w}$ in $\mathbb{V}$

(2) $T(\alpha \vec{v}) = \alpha T(\vec{v})$ for $\vec{v}$ in $\mathbb{V}$, α in $\mathbb{R}$.

- When $\mathbb{V}$ has finite dimension & basis $B = \{\vec{v}_1, \ldots, \vec{v}_p\}$ with $\dim \mathbb{V} = p$, we can understand T from the map $\tilde{T}: \mathbb{R}^p \longrightarrow \mathbb{W}$ via $\tilde{T}([\vec{v}]_B) = T(\vec{v})$

$$\begin{tikzcd} \mathbb{R}^p \arrow[r, "\tilde{T}"] & \mathbb{W} \\ \mathbb{V} \arrow[u, "\text{take coordinates with respect to } B"] \arrow[ur, "T"'] & \end{tikzcd}$$

§1. Null Space & Range of $T: \mathbb{V} \longrightarrow \mathbb{W}$:

- Next, we want to study the analog of Null Space & Range for $F: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ in this new framework. We define them in the same way!

Def 1: The Null Space of T is $\mathcal{N}(T) = \{\vec{v} \text{ in } \mathbb{V} : T(\vec{v}) = \mathbb{0}_{\mathbb{W}}\}$

Def 2: The Range of T is $\mathcal{R}(T) = \{\vec{w} \text{ in } \mathbb{W} : \text{we can find } \vec{v} \text{ in } \mathbb{V} \text{ with } T(\vec{v}) = \vec{w}\}$

(Range = image of the map T)

- Just as it happens for linear maps $\mathbb{R}^n \longrightarrow \mathbb{R}^m$ these 2 sets are vector subspaces of $\mathbb{V}$ & $\mathbb{W}$! More precisely:

Theorem 1: (1) $T(\mathbb{0}_{\mathbb{V}}) = \mathbb{0}_{\mathbb{W}}$

(2) $\mathcal{N}(T)$ is a subspace of $\mathbb{V}$

(3) $\mathcal{R}(T)$ ______ $\mathbb{W}$

Proof (1) We need to use $\mathbb{0}_{\mathbb{V}} = 0 \cdot \mathbb{0}_{\mathbb{V}}$ & $\mathbb{0}_{\mathbb{W}} = 0 \cdot \vec{w}$ for any $\vec{w}$ in $\mathbb{W}$

$$T(\mathbb{0}_{\mathbb{V}}) = T(0 \cdot \mathbb{0}_{\mathbb{V}}) \underset{\substack{\uparrow \\ T \text{ linear}}}{=} 0 \cdot \underbrace{T(\mathbb{0}_{\mathbb{V}})}_{=: \vec{w}} = \mathbb{0}_{\mathbb{W}}.$$

(2) Need to check $\mathcal{N}(T)$ satisfies the 3 properties for subspaces of $\mathbb{V}$

(S1) $\mathbb{0}_{\mathbb{V}}$ in $\mathcal{N}(T)$ true by (1)

(S2) If $\vec{v}, \vec{u}$ in $\mathcal{N}(T)$, then $\vec{v}+\vec{u}$ in $\mathcal{N}(T)$

Why? $T(\vec{v}) = \mathbb{0}_{\mathbb{W}}$, $T(\vec{u}) = \mathbb{0}_{\mathbb{W}}$ then $T(\vec{v}+\vec{u}) \overset{T \text{ linear}}{=} T(\vec{v}) + T(\vec{u}) = \mathbb{0}_{\mathbb{W}} + \mathbb{0}_{\mathbb{W}} = \mathbb{0}_{\mathbb{W}}$

so $\vec{v}+\vec{u}$ is also in $\mathcal{N}(T)$. ✓

(S3) If $\vec{v}$ in $\mathcal{N}(T)$ & α in $\mathbb{R}$, then $\alpha\cdot\vec{v}$ in $\mathcal{N}(T)$.

Why? $T(\vec{v}) = \mathbb{0}_{\mathbb{W}}$, so $T(\alpha\cdot\vec{v}) = \alpha\cdot T(\vec{v}) = \alpha\cdot\mathbb{0}_{\mathbb{W}} = \mathbb{0}_{\mathbb{W}}$ ✓

(3) Need to check that $\mathcal{R}(T)$ satisfies the 3 properties for subspaces of $\mathbb{W}$.

(S1) $\mathbb{0}_{\mathbb{W}}$ in $\mathcal{R}(T)$ because $\mathbb{0}_{\mathbb{W}} = T(\mathbb{0}_{\mathbb{V}})$.

(S2) If $\vec{w}_1, \vec{w}_2$ in $\mathcal{R}(T)$, then $\vec{w}_1 + \vec{w}_2$ in $\mathcal{R}(T)$

Why? We know $T(\vec{v}_1) = \vec{w}_1$ for some $\vec{v}_1$ in $\mathbb{V}$

$T(\vec{v}_2) = \vec{w}_2$ ——— $\vec{v}_2$ ———

Then $\vec{w}_1 + \vec{w}_2 = T(\vec{v}_1) + T(\vec{v}_2) \underset{T \text{ linear}}{=} T(\underbrace{\vec{v}_1+\vec{v}_2}_{=\vec{v} \text{ in } \mathbb{V}})$ as we wanted! ✓

(S3) If $\vec{w}$ in $\mathcal{R}(T)$ & α in $\mathbb{R}$, then $\alpha\cdot\vec{w}$ in $\mathcal{R}(T)$

Why? We have $\vec{w} = T(\vec{v})$ for some $\vec{v}$ in $\mathbb{V}$.

Then $\alpha\cdot\vec{w} = \alpha T(\vec{v}) \underset{T \text{ linear}}{=} T(\underbrace{\alpha\vec{v}}_{\vec{u} \text{ in } \mathbb{V}})$ as we wanted! ✓ $\square$

• Our next result says that we can compute $\mathcal{R}(T)$ very fast when $\mathbb{V}$ has finite dimension

Theorem 2: Fix $T: \mathbb{V} \longrightarrow \mathbb{W}$ linear transformation & assume $\dim \mathbb{V} = p$

Let $\mathcal{B} = \{v_1, \dots, v_p\}$ be a basis for $\mathbb{V}$. Then:

(1) $\mathcal{R}(T) = \mathrm{Sp}(T(v_1), \dots, T(v_p))$, so $\mathcal{R}(T)$ has dimension $\leq p$.

(2) $\mathcal{N}(T) = \{\mathbb{0}_{\mathbb{V}}\}$ if and only if $\{T(v_1), \dots, T(v_p)\}$ is l.i. in $\mathbb{W}$.

Proof: (1) Pick $\vec{w}$ in $\mathcal{R}(T)$ & fix $\vec{v}$ in $\mathbb{V}$ with $w = T(\vec{v})$

Since $\mathcal{B}$ is a basis for $\mathbb{V}$ we can write $\vec{v} = \alpha_1\vec{v}_1 + \dots + \alpha_p\vec{v}_p$, where $[\vec{v}]_{\mathcal{B}} = \begin{bmatrix}\alpha_1 \\ \alpha_p\end{bmatrix}$ in $\mathbb{R}^p$.

But then $\vec{w} = T(\vec{v}) = T(\alpha_1 \vec{v}_1 + \cdots + \alpha_p \vec{v}_p) = \alpha_1 \boxed{T(\vec{v}_1)} + \cdots + \alpha_p \boxed{T(\vec{v}_p)}$

Conclude: $\vec{w}$ in $Sp(T(\vec{v}_1), \ldots, T(\vec{v}_p))$ for any $\vec{w}$ in $R(T)$ $(*)$

- Since $T(\vec{v}_1), \ldots, T(\vec{v}_p)$ are all in $R(T)$ & $R(T)$ is a subspace of W

we have $Sp(T(v_1), \ldots, T(\vec{v}_p))$ lies in $R(T)$

From this & $(*)$ we get $R(T) = Sp(T(\vec{v}_1), \ldots, T(\vec{v}_p))$.

(2) We start by setting up a dependency relation for $\{T(\vec{v}_1), \ldots, T(\vec{v}_p)\}$ is l.i.

$(*)$ $\boxed{0_W = \alpha_1 T(\vec{v}_1) + \cdots + \alpha_p T(\vec{v}_p)} \underset{T \text{ linear}}{=} T(\alpha_1 \vec{v}_1 + \cdots + \alpha_p \vec{v}_p)$

This says $\alpha_1 \vec{v}_1 + \cdots + \alpha_p \vec{v}_p$ is in $N(T)$

- If $N(T) = \{0_V\}$ we get $0_V = \underline{\alpha_1} \vec{v}_1 + \cdots + \underline{\alpha_p} \vec{v}_p$

but $\{\vec{v}_1, \ldots, \vec{v}_p\}$ is l.i (it's a basis!) so this forces $\alpha_1 = \cdots = \alpha_p = 0$

Going back to the boxed expression $(*)$, we conclude $\{T(\vec{v}_1), \ldots, T(\vec{v}_p)\}$ is l.i.

- Conversely, assume $\{T(\vec{v}_1), \ldots, T(\vec{v}_p)\}$ is l.i. We want to show $N(T) = \{0_V\}$

To do so, we pick $\vec{v}$ in $N(T)$ & write it using the basis B, that is

$(*)$ $\vec{v} = \beta_1 \vec{v}_1 + \cdots + \beta_p \vec{v}_p$ for some $\beta_1, \ldots, \beta_p$ in $\mathbb{R}$

Now, we apply T to both sides:

$$\boxed{\vec{0}} \underset{\vec{v} \text{ in } N(T)}{=} T(\vec{v}) = T(\beta_1 \vec{v}_1 + \cdots + \beta_p \vec{v}_p) \overset{T \text{ linear}}{=} \boxed{\beta_1 T(\vec{v}_1) + \cdots + \beta_p T(\vec{v}_p)}$$

So $\vec{0} = \beta_1 \underbrace{T(\vec{v}_1)}_{} + \cdots + \beta_p \underbrace{T(\vec{v}_p)}_{}$ forces $\beta_1 = \cdots = \beta_p = 0$

l.i

So replacing in $(*)$ we get $\vec{v} = 0 \cdot \vec{v}_1 + 0 \cdot \vec{v}_2 + \cdots + 0 \cdot \vec{v}_p = 0_V$.

Conclude: The only element in $N(T)$ is 0_V. $\square$

Example: $T: P_2 \longrightarrow \mathbb{R}^2$ $\quad T(P) = \begin{bmatrix} P(1) \\ P'(1) \end{bmatrix}$ for P in P_2 is linear (HW 9)

$N(T) = \{P_{(x)} = a + bx + cx^2 : \begin{matrix} P(1) = a + b + c = 0 \\ P'(1) = \quad b + 2c = 0 \end{matrix}\}$ $\rightsquigarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$ REF, indep

$a = c$
$b = -2c$

$= \{c + (-2c)x + cx^2 : c \text{ in } \mathbb{R}\}$

$= \{c(1 - 2x + x^2) : c \text{ in } \mathbb{R}\} = Sp(1 - 2x + x^2)$ $\quad$ dim $= 1$

$R(T) = \left\{ \begin{bmatrix} a + b + c \\ b + 2c \end{bmatrix} : a, b, c \text{ in } \mathbb{R} \right\}$ $\quad$ dim $= 2 \leq 3 = \dim P_2$

$= \left\{ a \begin{bmatrix} 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\} = Sp\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right)$

$B = \{1, x, x^2\} \rightsquigarrow T(1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \; T(x) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \; T(x^2) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \rightsquigarrow$ as we found

• The next result relates injectivity of T & $\mathcal{N}(T) = \vec{0}_V$

Proposition: Fix $T: V \longrightarrow W$ linear transformation.

(1) $T(\vec{v}) = T(\vec{u})$ if and only if $\vec{u} - \vec{v}$ in $\mathcal{N}(T)$

(2) T is injective (meaning $T(\vec{v}) = T(\vec{u})$ forces $\vec{v} = \vec{u}$) if & only if $\mathcal{N}(T) = \{\vec{0}_V\}$

Proof: (2) follows easily from (1) because $\vec{u} - \vec{v} = \vec{0}_V$ means $\vec{v} = \vec{u}$.

To show (1), we use the linear properties of T:

$T(\vec{v}) = T(\vec{u})$ means $\boxed{\vec{0}_W} = T(\vec{u}) - T(\vec{v}) = T(\vec{u}) + (-1)\,T(\vec{v})$

$= T(\vec{u}) + T((-1)\vec{v}) = T(\vec{u} + (-1)\vec{v}) = \boxed{T(\vec{u} - \vec{v})}$

This means $\vec{u} - \vec{v}$ in $\mathcal{N}(T)$ $\square$

ASIDE OBSERVATION:

Q. Why is this relevant? We can thing of breaking (or "partition") V into different set (that do not intersect), according to their value under T

For each w in W we set $V_w = \{\vec{u} \text{ in } V : T(\vec{u}) = w\}$

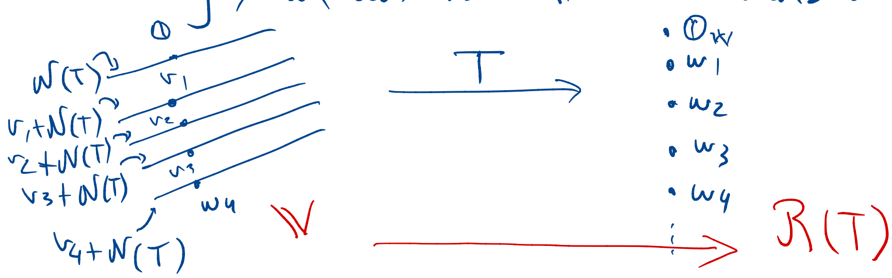
- If w is not in $\mathcal{R}(T)$, then $V_w = \emptyset$ (empty set)
- If w is in $\mathcal{R}(T)$, we know $\vec{w} = T(\vec{v})$ for some $\vec{v}$ in V

Prop (1) says that $V(w) = \vec{v} + \mathcal{N}(T)$

Why? Pick $\vec{u}$ in $V(w)$, then $T(\vec{u}) = T(\vec{v})$ so $(\vec{u} - \vec{v})$ in $\mathcal{N}(T)$

But then $\vec{u} = \vec{v} + \underbrace{(\vec{u} - \vec{v})}_{\text{in } \mathcal{N}(T)}$

Essentially, we can view V as "translations" of $\mathcal{N}(T)$ by vectors $\vec{v}$.



This is the essence of the rank-nullity theorem, as we will see next.

§3 Rank-Nullity Theorem

Before we state & prove the theorem, we start with an example

- Back to our example from page 3: $\mathbb{V} = \mathcal{P}_2$, $\mathbb{W} = \mathcal{R}(T) = \mathbb{R}^2$ $T: \mathcal{P}_2 \to \mathbb{R}^2$, $p \mapsto \begin{bmatrix} p(1) \\ p'(1) \end{bmatrix}$

$$\mathbb{V}_{\begin{bmatrix}1\\0\end{bmatrix}} = 1 + \mathcal{N}(T) = 1 + \alpha(1-2x+x^2) \quad \text{for } \alpha \text{ in } \mathbb{R}$$

$$\mathbb{V}_{\begin{bmatrix}1\\1\end{bmatrix}} = x + \mathcal{N}(T) = x + \beta(1-2x+x^2) \quad \text{for } \beta \text{ in } \mathbb{R}$$

Q: What about $\mathbb{V}_{\vec{w}}$ for other $\vec{w}$ in $\mathbb{R}^2$?

$$\mathbb{V}_{\vec{w}} = \vec{v} + \mathcal{N}(T)$$, but what is $\vec{v}$?

Use $\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\right\}$ is a basis for $\mathcal{R}(T)$ Write $\vec{w} = a\begin{bmatrix}1\\0\end{bmatrix} + b\begin{bmatrix}1\\1\end{bmatrix}$

So $\vec{w} = a\,T(1) + b\,T(x) = T(a+bx)$

Answer $\mathbb{V}_{a\begin{bmatrix}1\\0\end{bmatrix}+b\begin{bmatrix}1\\1\end{bmatrix}} = (a+bx) + \mathcal{N}(T)$

Claim: $\{\underbrace{1, x}_{\downarrow T \text{ basis for } \mathcal{R}(T)=\mathbb{R}^2}, \underbrace{\boxed{1-2x+x^2}}_{\text{basis for } \mathcal{N}(T);\ \downarrow\ \vec{0}_{\mathbb{W}}}\}$ is a basis for $\mathcal{P}_2$

$$\dim \mathcal{P}_2 = \dim \mathcal{R}(T) + \dim \mathcal{N}(T)$$
$$3 = 1 + 2$$

This is the rank-nullity theorem in a nutshell!

Def: Assume $\mathbb{V}$ is finite dimensional, & let $T: \mathbb{V} \to \mathbb{W}$ be a linear transformation.

- $\text{nullity}(T) = \dim \mathcal{N}(T)$ (finite & $\leq \dim \mathbb{V}$)
- $\text{rank}(T) = \dim \mathcal{R}(T)$ ($\leq \dim \mathbb{V}$ by Theorem 2)

Rank-Nullity Theorem: Assume $T: \mathbb{V} \to \mathbb{W}$ linear & $\dim \mathbb{V}$ finite

Then: $\text{rank}(T) + \text{nullity}(T) = \dim \mathbb{V}$.

Consequence: A $m \times n$ matrix $\text{rank}(A) + \text{nullity}(A) = n$ (# cols of A)

Proof of Consequence: A defines a linear map $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$
$$\vec{v} \longmapsto A\cdot\vec{v}$$

But $\mathcal{N}(A) = \mathcal{N}(T)$
$\mathcal{R}(A) = \mathcal{R}(T)$ (Lecture 23)

Rank-Nullity Thm says: $\underbrace{\dim \mathcal{N}(T)}_{=\text{nullity}(A)} + \underbrace{\dim \mathcal{R}(T)}_{=\text{rank}(A)} = n$

- Before discussing the proof of Rank-Nullity Theorem, we show a second example:
(optional reading, but very insightful)

Example 2: $T: M_{2\times 3} \longrightarrow \mathcal{P}_4$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \longmapsto (a_{12}+a_{23})x^4 + (2a_{22}+3a_{13})x^3 + (a_{11}-a_{23})$$

- T is linear because we have a linear map $\tilde{T}: M_{2\times 3} \longrightarrow \mathbb{R}^5$
$$A \longmapsto [T(A)]_B$$
after choosing the basis $B = \{1, x, x^2, x^3, x^4\}$ for $\mathcal{P}_4$.

Indeed: $[T(A)]_B = \begin{bmatrix} a_{11}-a_{23} \\ 0 \\ 0 \\ 2a_{22}+3a_{13} \\ a_{12}+a_{23} \end{bmatrix}$ only involves linear expressions in the coefficients of $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$.

- $\mathcal{N}(T) = ?$ Need $\begin{cases} a_{12} + a_{23} = 0 \\ 2a_{22} + 3a_{13} = 0 \\ a_{11} - a_{23} = 0 \end{cases}$ (This system also characterizes $\mathcal{N}(\tilde{T})$)

Must solve: $\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 3 & 0 & 2 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} a_{11} \\ a_{12} \\ a_{13} \\ a_{21} \\ a_{22} \\ a_{23} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 3 & 0 & 2 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 3 & 0 & 2 & 0 \end{bmatrix} \longrightarrow \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 2/3 & 0 \end{bmatrix}}_{\text{REF}}$$

indep vars

$a_{11} = a_{23}$
$a_{12} = a_{23}$
$a_{13} = -\frac{2}{3} a_{22}$

Any A in $\mathcal{N}(T)$ is $A = \begin{bmatrix} a_{23} & a_{23} & -\frac{2}{3}a_{22} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = a_{21}\underline{\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}} + a_{22}\underline{\begin{bmatrix} 0 & 0 & -2/3 \\ 0 & 1 & 0 \end{bmatrix}} + a_{23}\underline{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$

So $\mathcal{N}(T)$ has basis $\left\{ \underset{= E_{13}}{\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}}, \underset{= E_{22} - \frac{2}{3}E_{13}}{\begin{bmatrix} 0 & 0 & -2/3 \\ 0 & 1 & 0 \end{bmatrix}}, \underset{= E_{11}+E_{12}+E_{23}}{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \right\}$ so $\dim = 3$.

- $\mathcal{R}(T) = ?$ We know $B' = \{E_{11}, E_{12}, E_{13}, E_{21}, E_{22}, E_{23}\}$ basis for $M_{2\times 3}$ ensures $\mathcal{R}(T) = \text{Sp}(T(E_{11}), T(E_{12}), T(E_{13}), T(E_{21}), T(E_{22}), T(E_{23}))$

By Rank-Nullity: $\dim R(T) = \dim M_{2\times 3} - \dim \mathcal{N}(T) = 6-3=3$

So we know $R(T) \neq \mathcal{P}_4$ & a basis will be obtained by finding 3 li vectors among $\{T(E_{11}), \ldots, T(E_{23})\}$

$T(E_{11}) = 1$
$T(E_{12}) = x^4$
$T(E_{13}) = 3x^3$ $\rightsquigarrow$ Can pick $\{1, 3x^3, x^4\}$
$T(E_{21}) = 0$
$T(E_{22}) = 2x^3$
$T(E_{23}) = x^4-1$

$$R(T) = Sp(1, 3x^3, x^4)$$
$$= Sp(1, x^3, x^4)$$

We get information about $R(\tilde{T})$ from this; just take $[\ ]_B$ of these 3 vectors: (B = standard basis for $\mathcal{P}_4$)

$$R(\tilde{T}) = Sp([1]_B, [x^3]_B, [x^4]_B)$$
$$= Sp\left(\begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix}\right) = Sp(e_1, e_4, e_5) \text{ in } \mathbb{R}^5$$

This is consistent will the formula we had for $\tilde{T}(A)$ (2^{nd} & 3^{rd} entry were $=0$).

$\boxed{\text{Proof of Rank-Nullity Theorem:}}$ Fix $\dim \mathbb{V} = p$ & $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_p\}$ basis for $\mathbb{V}$

- We start by recalling a fact: $\text{rank}(T) \leq p$ (by Theorem 2)
- We prove 2 special cases, where $\text{rank}(T) = 0$ & $\text{rank}(T) = p$

(1) Assume $\text{rank}(T)=0$, meaning $R(T) = \{\mathbb{0}_{\mathbb{W}}\}$. In this case $T(\vec{v}) = \mathbb{0}_{\mathbb{W}}$ for all $\vec{v}$ so $\mathcal{N}(T) = \mathbb{V}$, & $\text{nullity}(T) = \dim \mathbb{V}$

Conclusion: $\text{rank}(T) + \text{nullity}(T) = 0 + \text{nullity}(T) = \dim \mathbb{V}$ ✓

(2) Assume $\text{rank}(T) = p$, meaning $R(T) = Sp(T(\vec{v}_1), \ldots, T(\vec{v}_p))$ has $\dim = p$

In particular $\{T(\vec{v}_1), \ldots, T(\vec{v}_p)\}$ is li, so by Theorem 2, $\mathcal{N}(T) = \{\mathbb{0}_{\mathbb{V}}\}$, nullity $=0$

Conclusion: $\text{rank}(T) + \text{nullity}(T) = \dim \mathbb{V} + 0 = \dim \mathbb{V}$ ✓

All that remains: is to prove the statement whenever $0 < \text{rank}(T) < p$

Names: $r = \text{rank}(T)$, $d = \text{nullity}(T)$ WANT To show: $r + d = p$

Since $r < p$, we know $\dim \text{Sp}(T(\vec{v}_1), \ldots, T(\vec{v}_p)) = r < p$ so $\{T(\vec{v}_1), \ldots, T(\vec{v}_p)\}$ is NOT li. Again, by Theorem 2 this means $\mathcal{N}(T) \neq \{\mathbb{0}_V\}$

In particular $0 < \text{nullity}(T) < p$ & we can pick a basis for $\mathcal{N}(T)$

Pick $\{\vec{w}_1, \ldots, \vec{w}_r\}$ basis for $\mathcal{R}(T)$
$\{\vec{v}_1, \ldots, \vec{v}_d\}$ ______ $\mathcal{N}(T)$

Write $\vec{w}_1 = T(\vec{u}_1)$
$\vec{w}_r = T(\vec{u}_r)$ for some $\vec{u}_1, \ldots, \vec{u}_r$ in V

Claim: $S = \{\vec{u}_1, \ldots, \vec{u}_r, \vec{v}_1, \ldots, \vec{v}_d\}$ is a basis for V

From here we get $p = r + d$. (total # of elements in any basis for V)

(1) S is li:

$$(*) \quad \mathbb{0}_V = \alpha_1 \vec{u}_1 + \cdots + \alpha_r \vec{u}_r + \beta_1 \vec{v}_1 + \cdots + \beta_d \vec{v}_d \qquad \text{Apply } T$$

$$\mathbb{0}_W = \alpha_1 T(\vec{u}_1) + \cdots + \alpha_r T(\vec{u}_r) + \beta_1 \mathbb{0}_W + \cdots + \beta_d \mathbb{0}_W$$

So $\mathbb{0}_W = \alpha_1 \vec{w}_1 + \cdots + \alpha_r \vec{w}_r$ But $\{\vec{w}_1, \ldots, \vec{w}_r\}$ is li

so $\boxed{\alpha_1 = \cdots = \alpha_r = 0}$

Now, replace this back in $(*)$ to get an equation only involving the β's.

$\mathbb{0}_V = \beta_1 \vec{v}_1 + \cdots + \beta_d \vec{v}_d$ but $\{\vec{v}_1, \ldots, \vec{v}_d\}$ is li so $\boxed{\beta_1 = \cdots = \beta_d = 0}$

Combine the boxed expressions to conclude S is li.

(2) S spans V:

Pick any $\vec{v}$ in V & apply T: Then $T(\vec{v})$ is in $\mathcal{R}(T)$, so we can write it as $T(\vec{v}) = \alpha_1 \vec{w}_1 + \cdots + \alpha_r \vec{w}_r$

$$= \alpha_1 T(\vec{u}_1) + \cdots + \alpha_r T(\vec{u}_r) = T(\alpha_1 \vec{u}_1 + \cdots + \alpha_r \vec{u}_r)$$

Conclude $\vec{v}$ & $\vec{u} = \alpha_1 \vec{u}_1 + \cdots + \alpha_r \vec{u}_r$ have the same image under T

By Proposition 1, $\vec{v} - \vec{u}$ lies in $\mathcal{N}(T)$.

Hence, we get $\vec{v} - \vec{u} = \beta_1 \vec{v}_1 + \cdots + \beta_d \vec{v}_r$

$$\rightsquigarrow \vec{v} = u + \beta_1 \vec{v}_1 + \cdots + \beta_d \vec{v}_d$$

$$= \alpha_1 \vec{u}_1 + \cdots + \alpha_r \vec{u}_r + \beta_1 \vec{v}_1 + \cdots + \beta_d \vec{v}_d$$

So $\vec{v}$ lies in $Sp(S)$

Conclusion: S spans W. $\square$

Upshot of the proof: The sets $W_{w_1}, \ldots, W_{w_r}$ describe W

$$W_{w_1} = u_1 + \mathcal{N}(T)$$

$$\vdots$$

$$W_{w_r} = u_r + \mathcal{N}(T)$$

(see aside Observation on page 4)

$$\boxed{W_{\alpha_1 \vec{w}_1 + \cdots + \alpha_r \vec{w}_r} = (\alpha_1 \vec{u}_1 + \cdots + \alpha_r \vec{u}_r) + \mathcal{N}(T)}$$

Next Time: Operations with linear transformations