Lecture XXVI : § 5.8 Operations with linear transformations

TODAY'S GOAL: Describe operations between linear transformations $T: \mathbb{V} \to \mathbb{W}$

Correspondence to keep in mind: $\boxed{T: \mathbb{R}^n \to \mathbb{R}^m \text{ linear} \iff A \text{ } m\times n \text{ matrix}}$

§1 Summary:

- $M_{m\times n}$, $\{T: \mathbb{R}^n \to \mathbb{R}^m : T \text{ linear}\}$ & $\{T: \mathbb{V} \to \mathbb{W} : T \text{ linear}\}$ are vector spaces (addition & scalar mult)
- We have multiplication / composition for some pairs of matrices / linear transformations

Operation	Matrices	$T: \mathbb{R}^n \to \mathbb{R}^m$ linear	$T: \mathbb{V} \to \mathbb{W}$ linear
(I) Addition	$A+C$ matrix $(A+C)_{ij} = A_{ij}+C_{ij}$	$F+G: \mathbb{R}^n \to \mathbb{R}^m$ linear $(F+G)(\vec{v}) = F(\vec{v})+G(\vec{v})$	$F+G: \mathbb{V} \to \mathbb{W}$ linear $(F+G)(\vec{v}) = F(\vec{v})+G(\vec{v})$
(II) Scalar Multiplication	$\alpha\cdot A$ matrix $(\alpha A)_{ij} = \alpha A_{ij}$	$\alpha\cdot T: \mathbb{R}^n \to \mathbb{R}^m$ $(\alpha\cdot T)(\vec{v}) = \alpha T(\vec{v})$	$\alpha\cdot T: \mathbb{R}^n \to \mathbb{R}^m$ $(\alpha\cdot T)(\vec{v}) = \alpha T(\vec{v})$
(III) Multiplication vs Composition	A in $M_{m\times n}$ C in $M_{s\times m}$ Then CA in $M_{s\times n}$	$F: \mathbb{R}^n \to \mathbb{R}^m$ linear $G: \mathbb{R}^m \to \mathbb{R}^s$ linear Then $G\circ F: \mathbb{R}^n \to \mathbb{R}^s$ linear $G\circ F(\vec{v}) = G(\underbrace{F(\vec{v})}_{\text{in } \mathbb{R}^m})$	$F: \mathbb{V} \to \mathbb{W}$ linear $G: \mathbb{W} \to \mathbb{U}$ linear Then $G\circ F: \mathbb{V} \to \mathbb{U}$ linear $G\circ F(\vec{v}) = G(\underbrace{F(\vec{v})}_{\text{in } \mathbb{W}})$

$\boxed{\begin{array}{l} F \text{ has matrix } A \\ G \text{ ——— } C \end{array}}$ (we saw this in Lecture 24)

§2 Examples: We show examples & highlight properties

EXAMPLE 1 $T_1: \mathcal{P}_2 \to \mathbb{R}$, $P(x) \mapsto P(1)$, $T_2: \mathcal{P}_2 \to \mathbb{R}$, $P(x) \mapsto P'(2)$ both are linear transf.

$T_1(a+bx+cx^2) = a+b+c$

$T_2(a+bx+cx^2) = b+2c(2) = b+4c$ (linear expressions in $[P]_{\{1,x,x^2\}} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$)

NEW FUNCTION: (by addition)

- $T_1+T_2: \mathcal{P}_2 \to \mathbb{R}$ is also linear
  $P(x) \mapsto P(1)+P'(2)$

In fact: $(T_1+T_2)(a+bx+cx^2) = a+b+c+b+4c = a+2b+5c$

- $\mathcal{N}(T_1+T_2) = \{ a+bx+cx^2 : a+2b+5c=0 \}$
  $= \{ (-2b-5c) + bx + cx^2 : b, c \in \mathbb{R} \} = \{ b(-2+x) + c(-5+x^2) \}$
  $= Sp(-2+x, -5+x^2)$ $\rightsquigarrow$ $\dim = 2$
- $\mathcal{N}(T_1) = \{ a+bx+cx^2 : a+b+c=0 \} = Sp((-1+x), (-1+x^2))$
- $\mathcal{N}(T_2) = \{ a+bx+cx^2 : b+4c=0 \} = Sp(1, -4+x^2)$

Observation: In general, we should <u>NOT</u> expect <u>any</u> relation between $\mathcal{N}(T_1)$, $\mathcal{N}(T_2)$ & $\mathcal{N}(T_1+T_2)$

- $\mathcal{R}(T_1+T_2)$ has dimension $= \dim \mathcal{P}_2 - \text{nullity}(T_1+T_2) = 3-2 = 1$

  Since $R(T_1+T_2)$ is a subspace of $\mathbb{R}$ & both have the same dimension, we conclude $R(T_1+T_2) = \mathbb{R}$

NEW FUNCTION: (by scalar multiplication)

$$3T_1 : \mathcal{P}_2 \longrightarrow \mathbb{R} \quad \text{is linear} \qquad (3T_1)_{(a+bx+cx^2)} = 3a+3b+3c$$
$$P \longmapsto 3P(1)$$

$$\mathcal{N}(3T_1) = \mathcal{N}(T) \quad \& \quad \mathcal{R}(3T_1) = \mathcal{R}(T_1)$$

Observation 2: In general: $\mathcal{N}(\alpha T) = \mathcal{N}(T)$ as long as $\underline{\underline{\alpha \neq 0}}$

$T: V \to W$ linear $\qquad \mathcal{R}(\alpha T) = \mathcal{R}(T)$

For $\underline{\underline{\alpha = 0}}$ we get $\mathcal{N}(0 \cdot T) = V$ & $\mathcal{R}(0 \cdot T) = \{ 0_W \}$

EXAMPLE 2

- $T_1 : M_{2\times 3} \longrightarrow \mathcal{P}_3$ linear
  $$A \longmapsto a_{11}x^3 + (a_{12}-a_{13})x^2 + a_{23}$$
- $T_2 : \mathcal{P}_3 \longrightarrow \mathbb{R}^2$ is linear
  $$a+bx+cx^2+dx^3 \longmapsto \begin{bmatrix} d-c \\ d-b+a \end{bmatrix}$$

NEW FUNCTION: T_2 composed with T_1

$$T_2 \circ T_1 : M_{2\times 3} \xrightarrow{T_2} \mathcal{P}_3 \xrightarrow{T_1} \mathbb{R}^2$$
$$A \longmapsto \underbrace{\boxed{a_{11}}}_{=d}x^3 + \underbrace{\boxed{(a_{12}-a_{13})}}_{=c}x^2 + \underbrace{\boxed{a_{23}}}_{=a} \; (b=0) \longmapsto \begin{bmatrix} a_{11} - (a_{12}-a_{13}) \\ a_{11} - 0 + a_{23} \end{bmatrix} = \begin{bmatrix} a_{11} - a_{12} + a_{13} \\ a_{11} + a_{23} \end{bmatrix}$$

linear in entries of A

In conclusion: $T_2 \circ T_1(A) = \begin{bmatrix} a_{11} - a_{12} + a_{13} \\ a_{11} + a_{23} \end{bmatrix}$

- $\underline{\text{Natural question 1}}$: Are $\mathcal{N}(T_1)$ & $\mathcal{N}(T_2 \circ T_1)$ related? Both are subspaces of $M_{2\times 3}$

$\underline{\text{Answer}}$: YES, they are!

$\vec{v}$ in $\mathcal{N}(T_1)$, then $T_1(\vec{v}) = \mathbb{0}$ in $\mathcal{P}_2$. Now apply T_2

$$(T_2 \circ T_1)(\vec{v}) = T_2(T_1(\vec{v})) = T_2(\mathbb{0}_{\mathcal{P}_2}) \underset{\uparrow}{=} \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

T_2 linear so $\mathbb{0} \to \mathbb{0}$.

$\underline{\text{conclude}}$: $\vec{v}$ in $\mathcal{N}(T_1 \circ T_2)$

$\boxed{\text{Observation 3:}}$ If $T_1 : V \longrightarrow W$, $T_2 : W \longrightarrow U$ linear, then all $\vec{v}$ in $\mathcal{N}(T_1)$ also lie in $\mathcal{N}(T_2 \circ T_1)$. $\underline{\text{In symbols}}$: $\boxed{\mathcal{N}(T_1) \subseteq \mathcal{N}(T_2 \circ T_1)}$

- $\underline{\text{Natural question 2}}$: Are $\mathcal{R}(T_2)$ & $\mathcal{R}(T_2 \circ T_1)$ related? Both are subspaces of $\mathbb{R}^2$

Pick $\vec{w}$ in $\mathcal{R}(T_2 \circ T_1)$. Then $\vec{w} = T_2 \circ T_1(\vec{v})$ for some $\vec{v}$ in $M_{2\times 3}$

$$= T_2(\underbrace{T_1(\vec{v})}_{\vec{u} \text{ in } \mathcal{P}_3}) \rightsquigarrow \text{ so } \vec{w} \text{ in } \mathcal{R}(T_2)$$

- Can we say something else?

$\mathcal{R}(T_2) = ?$

$$\mathcal{R}(T_2) = Sp(T_2(1), T_2(x), T_2(x^2), T_2(x^3))$$
$$= Sp\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \mathbb{R}^2$$

$\mathcal{R}(\underbrace{T_2 \circ T_1}_{=:T}) = ?$

$$\mathcal{R}(T_2 \circ T_1) = Sp(T(E_{11}), T(E_{12}), T(E_{13}), T(E_{21}), T(E_{22}), T(E_{23}))$$
$$= Sp\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \mathbb{R}^2$$

In this case: $\mathcal{R}(T_2) = \mathcal{R}(T) = \mathbb{R}^2$

$\boxed{\text{Observation 4:}}$ If $T_1 : V \longrightarrow W$, $T_2 : W \longrightarrow U$ linear, then all vectors in $\mathcal{R}(T_2 \circ T_1)$ also lie in $\mathcal{R}(T_2)$. In symbols: $\boxed{\mathcal{R}(T_2 \circ T_1) \subseteq \mathcal{R}(T_2)}$

Furthermore, if $\mathcal{R}(T_1) = W$, $\underline{\underline{\text{then}}}$ $\mathcal{R}(T_2 \circ T_1) = \mathcal{R}(T_2)$

- Next, we study surjective & invertible transformations:

§3. Surjective or Onto Transformations:

Def A linear transformation $T: \mathbb{V} \longrightarrow \mathbb{W}$ is onto (or surjective) if $\mathcal{R}(T) = \mathbb{W}$

Example 1: $T_2: \mathcal{P}_3 \longrightarrow \mathbb{R}^2$ (from Example 2) $T_2(a+bx+cx^2+dx^3) = \begin{bmatrix} d-c \\ d-b+a \end{bmatrix}$ is onto since $\mathcal{R}(T_2) = \mathbb{R}^2$

Example 2: $T: \mathcal{P}_3 \longrightarrow \mathcal{P}_2$ $T(P(x)) = P'(x)$ is onto.

Why? Fix $f(x) = a+bx+cx^2$ in $\mathcal{P}_2$ we pick $P(x) = \int_0^x f(t)\,dt = at + \frac{b}{2}t^2 + \frac{c}{3}t^3 \Big|_0^x$
$= ax + \frac{b}{2}x^2 + \frac{cx^3}{3}$

Then $T(P(x)) = f(x)$ by Fundamental Thm of Calculus

Alternative way to check this? $\mathcal{R}(\mathcal{P}_3) = Sp(\underset{0}{T(1)}, \underset{1}{T(x)}, \underset{2x}{T(x^2)}, \underset{3x^2}{T(x^3)})$
$= Sp(1, 2x, 3x^2) = Sp(1, x, x^2) = \mathcal{P}_2$

Example 3: $\tilde{T}: \mathcal{P}_3 \longrightarrow \mathcal{P}_3$ $\tilde{T}(P(x)) = P'(x)$ is NOT onto because $\mathcal{R}(\tilde{T}) = \mathcal{P}_2$
(same formula as T from Example 2) $\mathcal{R}(T)$

Q: What can we use to check if $\mathcal{R}(T) = \mathbb{W}$? A Dimension!

Proposition: If $T: \mathbb{V} \longrightarrow \mathbb{W}$ is linear & dimension of $\mathbb{W}$ is finite, then we can check if T is onto by checking if $\dim \mathcal{R}(T) = \dim \mathbb{W}$.

§4. Invertible Transformations

This notion combines both the notion of invertible matrices & maps $f: \mathbb{R} \longrightarrow \mathbb{R}$ that are bijections.

Def: A linear transformation $T: \mathbb{V} \longrightarrow \mathbb{W}$ is invertible if we can find another linear transformation $S: \mathbb{W} \longrightarrow \mathbb{V}$ satisfying:

(1) $S \circ T: \mathbb{V} \longrightarrow \mathbb{V}$, $\vec{v} \longmapsto \vec{v}$ & (2) $T \circ S: \mathbb{W} \longrightarrow \mathbb{W}$, $\vec{w} \longmapsto \vec{w}$.

(so $S \circ T$ = identity map on $\mathbb{V}$) (so $T \circ S$ = identity map on $\mathbb{W}$)

Observation: If T is invertible, we can only find one S with these properties.

Because of this, we can write $S = T^{-1}$ (like we did when finding inverses of (square) non-singular matrices)

<u>Alternative name</u>: An invertible linear transformation is also called an <u>isomorphism</u>

<u>Why?</u> It basically says that for all practical purposes $\mathbb{V}$ & $\mathbb{W}$ can be thought of as the <u>same</u> vector space (E.g. $T: \mathcal{P}_2 \longrightarrow \mathbb{R}^3$ is an isomorphism)
$$a+bx+cx^2 \longmapsto \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

<u>Special case</u>: $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ linear map associated to an $m \times n$ matrix A
$$\vec{v} \longmapsto A\vec{v}$$

Then: T is invertible if and only if $m=n$ & A is an invertible matrix

Furthermore: $T^{-1}: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is the linear map associated to A^{-1}.
$$\vec{w} \longmapsto A^{-1}\vec{w}$$

<u>Why?</u> (1) $T^{-1} \circ T(\vec{v}) = T^{-1}(A\vec{v}) = A^{-1}(A\vec{v}) = (A^{-1}A)\vec{v} = \mathrm{Id}\,\vec{v} = \vec{v}$ for all $\vec{v}$

(2) $T \circ T^{-1}(\vec{w}) = T(A^{-1}\vec{w}) = A(A^{-1}\vec{w}) = (AA^{-1})\vec{w} = \mathrm{Id}\,\vec{w} = \vec{w}$ for all $\vec{w}$

Main Example Pick $\mathbb{V}$ a finite-dimensional vector space & fix a basis $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_p\}$ for $\mathbb{V}$

Then:
$$\boxed{\begin{aligned} T: \mathbb{V} &\longrightarrow \mathbb{R}^p \\ \vec{v} &\longmapsto [\vec{v}]_{\mathcal{B}} \end{aligned}}$$
is an <u>invertible</u> linear transformation

<u>Why?</u> Propose a formula for an inverse!

Set $S: \mathbb{R}^p \longrightarrow \mathbb{V}$ it is linear by construction.
$$\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_p \end{bmatrix} \longmapsto \alpha_1\vec{v}_1 + \cdots + \alpha_p\vec{v}_p$$

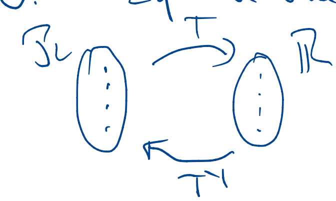
(1) $S \circ T(\vec{v}) = S\left(\underbrace{\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_p \end{bmatrix}}_{=[\vec{v}]_{\mathcal{B}}}\right) = \alpha_1\vec{v}_1 + \cdots + \alpha_p\vec{v}_p = \vec{v}$

(2) $T \circ S\left(\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_p \end{bmatrix}\right) = T(\alpha_1\vec{v}_1 + \cdots + \alpha_p\vec{v}_p) = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_p \end{bmatrix}$ by definition of $[\ ]_{\mathcal{B}}$.

<u>Conclude</u>: $S = T^{-1}$

Non-examples ① $T: \mathcal{P}_2 \longrightarrow \mathbb{R}$ is <u>NOT</u> invertible
$$p \longmapsto p(1)$$

<u>Issue</u>: 0 and $1-x$ both map to 0. If we could write T^{-1}, what should be the polynomial $T^{-1}(0)$?



In particular: $1-x \underset{\substack{\uparrow \\ \text{if } T^{-1} \text{ exists}}}{=} T^{-1} \circ T(1-x) = T^{-1}(\mathbb{0}) \underset{\substack{\uparrow \\ T^{-1} \text{ is linear}}}{=} \mathbb{0}$ but $1-x$ is NOT the $\mathbb{0}$ of P_2!

Problem: $N(T) \neq \{\mathbb{0}_V\}$ (Equivalently: T is NOT injective)

2) $T: \mathbb{R} \longrightarrow \mathbb{R}^2$ is linear but NOT invertible
$x \longmapsto \begin{bmatrix} 2x \\ x \end{bmatrix}$

Why? $R(T)$ = line through $(0,0)$ with direction $\binom{2}{1}$

So $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ is not in $R(T)$. What should be $\boxed{T^{-1}\left(\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right)}$ in $\mathbb{R}$? Call this # by a

In particular $\begin{bmatrix} 3 \\ 1 \end{bmatrix} \underset{\substack{\uparrow \\ \text{If } T^{-1} \text{ exists}}}{=} T \circ T^{-1}\left(\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right) = T(a) = \begin{bmatrix} 2a \\ a \end{bmatrix} = a \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

But: $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ & $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ are not parallel, so we can never have a number a verifying $\begin{bmatrix} 3 \\ 1 \end{bmatrix} = a \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Conclude: T^{-1} cannot exist.

Problem: $R(T) \neq \mathbb{R}^2$ (T was not surjective)

These two last examples give insight into the following question:

Q: How can we determine if a linear transformation $T: V \longrightarrow W$ is invertible?

The next proposition gives some easy things to check:

Proposition: If $T: V \longrightarrow W$ linear is invertible then:

(1) $N(T) = \{\mathbb{0}_V\}$ (T is injective)

(2) $R(T) = W$ (T is surjective)

Proof: We saw the main ideas in the examples above. But more precisely:

(1) We know $T^{-1} \circ T: V \longrightarrow V$, $\vec{v} \longmapsto \vec{v}$ so $N(T^{-1} \circ T) = \{\mathbb{0}_V\}$

But we know from $\boxed{\text{Observation 3}}$ that $N(T) \subseteq N(T^{-1} \circ T) = \{\mathbb{0}_V\}$

So we conclude $N(T)$ can only contain $\mathbb{0}_V$.

(2) We know $T \circ T^{-1}: W \longrightarrow W$, $\vec{w} \longmapsto \vec{w}$ so $R(T \circ T^{-1}) = W$

But we know from $\boxed{\text{Observation 4}}$ that $\underbrace{R(T\circ T^{-1})}_{=\mathbb{W}} \subseteq R(T)$

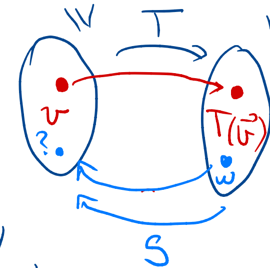
So all vectors $\vec{w}$ in $\mathbb{W}$ lie in $R(T)$. Since $R(T)$ is a subspace of $\mathbb{W}$ we conclude $R(T)=\mathbb{W}$. □

- It turns out these 2 things are enough!

Theorem 1: If $T:\mathbb{V}\longrightarrow\mathbb{W}$ is linear with $N(T)=\{0_{\mathbb{V}}\}$ & $R(T)=\mathbb{W}$ then T is invertible.

Proof: We want to define the inverse to T:

$$S:\mathbb{W}\longrightarrow\mathbb{V}$$
$$\vec{w}\longmapsto ?$$



Once S is defined, we must check: S is linear & $\begin{cases} S\circ T(\vec v)=\vec v & \text{for all } \vec v \text{ in } \mathbb{V}\\ T\circ S(\vec w)=\vec w & \text{for all } \vec w \text{ in } \mathbb{W}\end{cases}$

- Since $R(T)=\mathbb{W}$, we can find $\vec v$ in $\mathbb{V}$ with $T(\vec v)=\vec w$. But since $N(T)=\{0_{\mathbb{V}}\}$ there is no other $\vec u$ with $T(\vec u)=\vec w$.

So, we define $S(\vec w)=\vec v$ if $T(\vec w)=\vec v$. (our hands are tied!)

By design: $S\circ T(\vec v)=S(\underbrace{T(\vec v)}_{=\vec w})=\vec v$ for all $\vec v$ in $\mathbb{V}$ ✓
by definition of S.

- S is linear:

(1) If $\vec w_1, \vec w_2$ in $\mathbb{W}$, pick $\vec v_1, \vec v_2$ in $\mathbb{V}$ with $T(\vec v_1)=\vec w_1 \rightsquigarrow S(\vec w_1)=\vec v_1$
$T(\vec v_2)=\vec w_2 \rightsquigarrow S(\vec w_2)=\vec v_2$

Note $T(\vec v_1+\vec v_2)=\vec w_1+\vec w_2 \rightsquigarrow S(\vec w_1+\vec w_2)=\vec v_1+\vec v_2$

Conclude: $S(\vec w_1+\vec w_2)=S(\vec w_1)+S(\vec w_2)$ ✓

(2) If $\vec w$ in $\mathbb{W}$ & α in $\mathbb{R}$, pick $\vec v$ in $\mathbb{V}$ with $T(\vec v)=\vec w \rightsquigarrow S(\vec w)=\vec v$

Then $T(\alpha\cdot\vec v)=\alpha T(\vec v)=\alpha\vec w \rightsquigarrow S(\alpha\vec w)=\alpha\vec v$)

Conclude: $S(\alpha\vec w)=\alpha S(\vec w)$. ✓

- Only missing thing to check: $T\circ S(\vec w)=\vec w$ for all $\vec w$ in $\mathbb{W}$

But again: $T(\underbrace{S(\vec w)}_{=\vec v})=T(\vec v)=\vec w$ by definition of S ✓

Special situation: $\dim \mathbb{V}$ is finite & $\dim \mathbb{V} = \dim \mathbb{W}$

Then: $\mathcal{N}(T) = \{\mathbb{0}_{\mathbb{V}}\}$ if and only if $\mathcal{R}(T) = \mathbb{W}$ by the Rank-Nullity Thm

Theorem 2: If $\dim \mathbb{V} = \dim \mathbb{W} = p$ then $T: \mathbb{V} \longrightarrow \mathbb{W}$ linear map is invertible if & only if $\mathcal{N}(T) = \{\mathbb{0}_{\mathbb{V}}\}$

Example: $\dim \mathbb{V} = p$, $\mathbb{W} = \mathbb{R}^p$, $\mathcal{B}$ basis for $\mathbb{V}$.
$T: \mathbb{V} \longrightarrow \mathbb{R}^p$, $\vec{v} \longmapsto [\vec{v}]_{\mathcal{B}}$ is invertible

because $\mathcal{N}(T) = \{\mathbb{0}_{\mathbb{V}}\}$.

(This was our $\boxed{\text{Main Example}}$ on page 5)

- Most notably, if $\dim \mathbb{V} = \dim \mathbb{W} = p$ we can always build an invertible linear transformation $T: \mathbb{V} \longrightarrow \mathbb{W}$

How? Pick $\mathcal{B}_1 = \{\vec{v}_1, \ldots, \vec{v}_p\}$ basis for $\mathbb{V}$

$\mathcal{B}_2 = \{\vec{w}_1, \ldots, \vec{w}_p\}$ —— $\mathbb{W}$

We define T by $\begin{cases} T(\vec{v}_1) = \vec{w}_1 \\ \vdots \\ T(\vec{v}_p) = \vec{w}_p \end{cases}$ since $\mathcal{B}_1$ is a basis, this assignment determines T uniquely ("Extend linearly using these assignments")

(How? $\boxed{T(\vec{v})} = T(\alpha_1 \vec{v}_1 + \cdots + \alpha_p \vec{v}_p) = \alpha_1 T(\vec{v}_1) + \cdots + \alpha_p T(\vec{v}_p)$

Write $[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_p \end{bmatrix}$ $= \boxed{\alpha_1 \vec{w}_1 + \cdots + \alpha_p \vec{w}_p}$)

But by construction $\mathcal{R}(T) = \mathbb{W}$ so we automatically get

$\dim \mathcal{N}(T) = \dim \mathbb{V} - \dim \mathcal{R}(T) = p - p = 0$ so $\mathcal{N}(T) = \{\mathbb{0}_{\mathbb{V}}\}$

By Theorem 2, this map T is invertible.

In symbols (Construction of T)

$$\begin{tikzcd} \mathbb{V} \arrow[r, "F"] \arrow[d, "T"'] & \mathbb{R}^p \arrow[d, equal] \\ \mathbb{W} \arrow[r, "G"] & \mathbb{R}^p \end{tikzcd}$$

$F(\vec{v}) = [v]_{\mathcal{B}_1}$ invertible

$G(\vec{w}) = [\vec{w}]_{\mathcal{B}_2}$ invertible

Our map is $T = G^{-1} \circ F : \mathbb{V} \to \mathbb{R}^p \to \mathbb{W}$. (Composition of invertible maps is invertible).