Lecture XXXI: §4.1–2 The eigenvalue problem for 2x2 matrices

§1 The eigenvalue (EV) problem for n×n matrices

- The name comes from German since "Eigen" = "self".
- The EV problem can be stated as follows:

Fix an $n\times n$ matrix A. We are interested in finding those lines in $\mathbb{R}^n$ through the origin (that is spaces of the form $Sp\langle \vec{w}\rangle$ for $\vec{w}\neq \vec{0}$) that are invariant under the linear transformation $T:\mathbb{R}^n \longrightarrow \mathbb{R}^n$, $\vec{v}\longmapsto A\vec{v}$.

In symbols, we want to find $\vec{w}\neq\vec{0}$ so that $\boxed{T(Sp\langle\vec{w}\rangle) \text{ lies in } Sp\langle \vec{w}\rangle}$

We need to find $\vec{w}\neq\vec{0}$ so $A\cdot\vec{w}$ lies in $Sp\langle\vec{w}\rangle$

EV Problem (version 1): Find $\vec{w}\neq\vec{0}$ satisfying $A\vec{w}=\lambda\vec{w}$ for some λ in $\mathbb{R}$

Names: $\vec{w}$ eigenvector, λ = eigenvalue

Equivalently, rather than looking for $\vec{w}$, we can look for the scalars λ

EV Problem: Find λ in $\mathbb{R}$ such that $A\begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix}=\lambda\begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix}$ has a non-trivial solution $\begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix}\neq\vec{0}$.

- λ = an eigenvalue of A (we can have more than one!)
- Solutions $\begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix}$ = eigenvectors of the eigenvalue λ (these were the $\vec{w}$'s we had earlier)

Example 1: $\lambda=0$ is an eigenvalue for A if and only if $\mathcal{N}(A)\neq\{\vec{0}\}$

Why? Want to have non-trivial solutions to $A\begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix}=\vec{0}$.

Example 2: $A=Id$ has only one eigenvalue $\lambda=1$. All $\vec{w}\neq\vec{0}$ are eigenvectors

§ 2 Motivation:

Eigenvalues have many applications (in Math an beyond).

For example:

(1) Solve differential equations

(2) Analyze population growth

(3) Calculate powers of matrices (much faster than usual)

(4) Simplify and draw conics in the plane (see Lecture 5)

$$ax^2+by^2+cxy+dx+ey+f=0 \quad \text{for } \underline{\text{fixed}} \text{ parameters } a,b,c,d,e,f \text{ in } \mathbb{R}$$

(5) Diagonalize linear transformations $T:\mathbb{R}^n \longrightarrow \mathbb{R}^n$

Let's discuss (5) in more detail (more on §4.7)

Say we have a basis B for $\mathbb{R}^n$ consisting of eigenvectors $\vec{w}_1, \dots \vec{w}_n$ for a matrix A. Then, $A\vec{w}_j = \lambda_j \vec{w}_j$ for some λ_j in $\mathbb{R}$

Then, the linear transformation $T: \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $\vec{v} \longmapsto A\vec{v}$ has a nice matrix representation with respect to the basis B, namely:

$$[T]_{BB} = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \quad \text{vs} \quad [T]_{EE} = A \quad E=\{e_1,\dots,e_n\}$$

It is a <u>diagonal matrix</u>. These are the simplest kind of linear transformations!

<u>Conclusion</u>: To "diagonalize" a linear transformation $T:\mathbb{R}^n \longrightarrow \mathbb{R}^n$ we need to find the eigenvalues of A & make sure we have a <u>basis</u> of eigenvectors. This is not always possible! (Otherwise the theory of linear transformations will be extremely easy!)

Example: $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ has one eigenvector $\lambda = 1$. & a line of eigenvectors $= \{\mu\, e_1$ for $\mu \neq 0.\}$

Why? $A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_2 \end{bmatrix} \overset{?}{=} \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \rightsquigarrow \begin{cases} x_1 + x_2 = \lambda x_1 \\ x_2 = \lambda x_2 \end{cases}$

We look at 2nd equation $x_2 - \lambda x_2 = 0$

$(1-\lambda)\, x_2 = 0 \longrightarrow 1-\lambda = 0$ or $\boxed{x_2 = 0}$

$\boxed{\lambda = 1}$

Case 1: $\lambda = 1$ then look at 1st equation:

$x_1 + x_2 = \lambda\, x_1 = x_1$ forces $x_2 = 0$

So $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ multiple for e_1

Note since $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ cannot be $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ & we already know $x_2 = 0$, then $x_1 \neq 0$.

This gives the answer we wanted

Case 2: $x_2 = 0$, then look at 1st equation:

$x_1 + x_2 = x_1 + 0 = \lambda x_1$ forces $x_1 = 0$ or $\lambda = 1$

But again, $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ forces $x_1 \neq 0$, so $\lambda = 1$

In conclusion, in both cases we have $\lambda = 1$ is the unique eigenvalue & all eigenvectors are of the form μe_1 for $\mu \neq 0$

So we don't have a basis for $\mathbb{R}^2$ consisting of eigenvectors (because any 2 vectors of the form μe_1, $\mu' e_1$ are linearly dependent!)

So A cannot be diagonalized!

Question: Why did we exclude $\vec{w} = \vec{0}$ when looking for eigenvectors?

Because $A\vec{0} = \vec{0} = \lambda \cdot \vec{0}$ works for every λ! We would have too many eigenvalues!!!

§3 Strategies for solving the EV Problem:

Let's fix A of size $n\times n$

EV Problem: Find λ in $\mathbb{R}$ with a nontrivial $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \neq \vec{0}$ solution to $A \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

Equivalently: $\underbrace{A}_{\text{matrix}} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} - \underbrace{\lambda}_{\text{scalar}} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$

To take $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ as a "common factor" we must replace λ with λI_n

(Why? $\lambda \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \lambda x_1 \\ \vdots \\ \lambda x_n \end{bmatrix}$ & $\lambda I_n \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \lambda & & 0 \\ & \ddots & \\ 0 & & \lambda \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \lambda x_1 \\ \vdots \\ \lambda x_n \end{bmatrix}$, so same answer!)

So $\vec{0} = A \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} - \lambda \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = A \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} - \lambda I_n \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = (A - \lambda I_n) \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ & $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \neq \vec{0}$

In particular $\vec{0} \neq \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ lies in $\mathcal{N}(A - \lambda I_n)$, so $A - \lambda I_n$ must be singular (as a matrix)

> Conclusion: EV Problem is equivalent to finding λ so that the $n\times n$ matrix $A - \lambda I_n$ is singular.

But we can use determinant to decide when a square matrix is singular

> STRATEGY TO SOLVE EV Problem

(1) Find all λ's where $\det(A - \lambda I_n) = 0$ (at most n of them) because $\det(A - \lambda I_n)$ will be a polynomial in λ of degree $= n$

(2) For any λ computed in (1), find the $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \neq \vec{0}$ solving $(A - \lambda I_n) \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ using Gauss-Jordan elimination.

Def: $E_\lambda = \mathcal{N}(A - \lambda I_n) =$ eigenspace of the eigenvalue λ (it's a subspace of $\mathbb{R}^n$ by definition!)

(any $\vec{w}$ in E_λ with $\vec{w} \neq \vec{0}$ will be an eigenvector for λ.)

We will see this strategy in action for 2x2 & 3x3 matrices.

§4 The EV Problem for 2x2 matrices:

Write $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then we compute $\det(A-\lambda I_2)=0$ & solve for λ

- $A-\lambda I_2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix}$
- $\det(A-\lambda I_2) = (a-\lambda)(d-\lambda) - bc$

(distribute) $= ad - a\lambda - d\lambda + \lambda^2 - bc$

(write it as a polynomial in λ) $= \boxed{\lambda^2} + (-a-d)\boxed{\lambda} + \underbrace{(ad-bc)}_{=\det(A)}$

Q: How to solve $\lambda^2 + p\lambda + q = 0$ for λ? Here, $p=-a-d$ $q=\det(A)$

A Use quadratic formula! $\boxed{\lambda = \dfrac{-p \pm \sqrt{p^2-4q}}{2}}$ Solutions to EV problem!

Observation:

- Solutions are real numbers if and only if $\boxed{p^2-4q>0}$
- One double solution (and real!) means $p^2-4q=0$
- If $p^2-4q<0$, then we'll have complex solutions (§4.6)

Let's see this in examples:

$\boxed{\text{EXAMPLE 1}}$ $A = \begin{bmatrix} 5 & -1 \\ 8 & -1 \end{bmatrix}$ Find eigenvalues & basis for Eigenspaces.

- $\det(A-\lambda I_2) = \lambda^2 + (-5+1)\lambda + (-5+8) = \lambda^2 - 4\lambda + 3 = 0$

Solutions: $\lambda = \dfrac{4 \pm \sqrt{16-4\cdot 3}}{2} = \dfrac{4 \pm \sqrt{4}}{2} = \dfrac{4 \pm 2}{2}$ So $\boxed{\lambda = 3 \quad \& \lambda = 1}$ are the 2 solutions

- $E_1 = \mathcal{N}(A - Id)$ & $E_3 = \mathcal{N}(A - 3Id)$ Two eigenspaces

Basis? • $A - Id = \begin{bmatrix} 4 & -1 \\ 8 & -2 \end{bmatrix} \xrightarrow[R_2 \to R_2 - 2R_1]{} \begin{bmatrix} 4 & -1 \\ 0 & 0 \end{bmatrix} \xrightarrow[R_1 \to \frac{R_1}{4}]{} \begin{bmatrix} 1 & -1/4 \\ 0 & 0 \end{bmatrix}$ REF indep var.

$x_1 = \frac{1}{4}x_2$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1/4 \\ 1 \end{bmatrix}$

So basis for E_1 is $\left\{ \begin{bmatrix} 1/4 \\ 1 \end{bmatrix} \right\}$

- $A-3I_2 = \begin{bmatrix} 5-3 & -1 \\ 8 & -1-3 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 8 & -4 \end{bmatrix} \xrightarrow[R_2 \to R_2 - 4R_1]{} \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 0 \end{bmatrix}$ REF, indep var

so $\begin{bmatrix} x_1 \\ x \end{bmatrix} = \begin{bmatrix} \frac{1}{2}x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$ is the form of any vector in $\mathcal{N}(A-3I_2)$

$\leadsto$ Basis for $E_3 = \left\{ \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} \right\}$

Obs: In this example $\left\{ \begin{bmatrix} 1/4 \\ 1 \end{bmatrix}, \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} \right\}$ is a basis of eigenvectors

$\boxed{\text{EXAMPLE 2}}$ $A = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$ Find eigenvalues & eigenspaces

Can guess $\lambda = 3$, $\lambda = -1$ & $E_3 = Sp\langle e_1 \rangle$, $E_{-1} = Sp\langle e_2 \rangle$ (because A is a diagonal matrix!)

Let's check this with the strategy we discussed earlier:

$\det(A - \lambda I_2) = \det\begin{pmatrix} 3-\lambda & 0 \\ 0 & -1-\lambda \end{pmatrix} \underset{\text{matrix is triangular}}{=} (3-\lambda)(-1-\lambda)$

So only solutions in λ to $(3-\lambda)(-1-\lambda)=0$ are $\lambda=-1$ & $\lambda=3$.

$E_3 = \mathcal{N}(A-3I_2) = \mathcal{N}\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} = Sp(e_1)$ is clear. (x_1 indep var, $x_2=0$.)

$E_{-1} = \mathcal{N}(A-(-1)I_2) = \mathcal{N}\begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix} = Sp(e_2)$ —— . ($x_1=0$, $x_2=$ indep var)

§5 An example for 3x3 matrices:

Ex $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -2 \\ 0 & 1 & -2 \end{bmatrix}$ Need to compute $\det(A-\lambda I_3)=0$ & find solns in λ.

$$A - \lambda I_3 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -2 \\ 0 & 1 & -2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = \begin{bmatrix} 1-\lambda & 1 & 0 \\ 0 & 1-\lambda & -2 \\ 0 & 1 & -2-\lambda \end{bmatrix}$$

We expand along the 1st column, since it has only 1 nonzero entry

$$\det(A-\lambda I_3) = (-1)^{1+1}(1-\lambda)\begin{vmatrix} 1-\lambda & -2 \\ 1 & -2-\lambda \end{vmatrix} = (1-\lambda)\left((1-\lambda)(-2-\lambda)+4\right) =$$

$$= (1-\lambda)\left(\lambda^2 + (-1-(-2))\lambda + (-2+2)\right) = (1-\lambda)(\lambda^2+\lambda) = (1-\lambda)\lambda(\lambda+1)$$

From here we get 3 eigenvalues $\lambda=0$, $\lambda=1$, $\lambda=-1$

$$\underline{E_0} = \mathcal{N}(A) = \mathcal{N}\left(\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -2 \\ 0 & 1 & -2 \end{bmatrix}\right)$$

indep var

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -2 \\ 0 & 1 & -2 \end{pmatrix} \xrightarrow[R_3 \to R_3 - R_2]{} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow[R_1 \to R_1 - R_2]{} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

REF

$x_1 = -2x_3$

$x_2 = 2x_3$

$$\rightsquigarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_3 \\ 2x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$$

$$\text{So } E_0 = Sp\left(\begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}\right)$$

$$\underline{E_1}: \mathcal{N}(A - I_2) = \mathcal{N}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -2 \\ 0 & 1 & -3 \end{pmatrix}$$

Rank of matrix is 2

So null-space has dim $= 3-2=1$

We can see e_1 is in the null-space

By dim reasons, we conclude $E_1 = Sp(e_1)$.

$$E_{-1} \quad \mathcal{N}(A+I_2) = \mathcal{N}\left(\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & -2 \\ 0 & 1 & -1 \end{bmatrix}\right)$$

indep var

$$\begin{pmatrix} 2 & -1 & 0 \\ 0 & 2 & -2 \\ 0 & 1 & -1 \end{pmatrix} \xrightarrow[R_2 \to R_2 - 2R_3]{} \begin{pmatrix} 2 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & -1 \end{pmatrix} \xrightarrow[R_1 \to R_1 + R_3]{} \begin{pmatrix} 2 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 1 & -1 \end{pmatrix} \xrightarrow[R_1 \to \frac{R_1}{2},\ R_2 \leftrightarrow R_3]{} \begin{pmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

REF

So $x_1 = -\frac{1}{2}x_3$

$x_2 = x_3$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 1 \end{bmatrix}$$

$$\text{So } E_{-1} = Sp\left(\begin{bmatrix} -\frac{1}{2} \\ 1 \\ 1 \end{bmatrix}\right)$$

$$\underline{\text{Check}}: \ A\begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0\begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}, \quad A\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad A\begin{bmatrix} -\frac{1}{2} \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -1 \\ -1 \end{bmatrix} = -1\begin{bmatrix} -\frac{1}{2} \\ 1 \\ 1 \end{bmatrix}$$

(The definition for being eigenvalues & eigenvectors is satisfied!)