

Lecture 36: §4.7 Similarity & Diagonalization

Last time: Defined the \mathbb{C} -vector space \mathbb{C}^n

- Studied complex eigenvectors of matrices of size $n \times n$ with real entries
- Real $n \times n$ symmetric matrices have ONLY real eigenvalues.

TODAY: Study the notion of similarity & diagonalization of matrices, that is, to find a basis of eigenvectors for \mathbb{R}^n . Main example: real symmetric $n \times n$ matrices

§1. Similarity:

Definition: Two matrices A & C of size $n \times n$ with complex entries are similar if there is a nonsingular $n \times n$ matrix S with complex entries such that $C = S^{-1}AS$

• For the applications we have in mind, we will only consider the case where A, C & S have real entries, but the definition is more general.

Motivation: A & C will be representing matrices of the SAME linear transformation

$T: \mathbb{R}^n \rightarrow \mathbb{R}^n$ with respect to two different bases: B_A & B_C & S will be the matrix recording the "change of basis" from B_A to B_C .

Typical situation: $T(\vec{v}) = A\vec{v}$ so $B_A = E = \{e_1, \dots, e_n\}$ canonical basis for \mathbb{R}^n

If $B_C = \{\vec{v}_1, \dots, \vec{v}_n\}$, then $S = [v_1 \dots v_n]$ (columns = vectors in the basis B_C)

• We will show that similar matrices have the same set of eigenvalues (with the same algebraic multiplicity). To this end, we show

Proposition 1: Similar matrices have the same characteristic polynomial

Proof: $P_C(t) = \det(S^{-1}AS - tI_n) = \det(S^{-1}(A - tI_n)S)$
 $(C = S^{-1}AS) = \det(S^{-1}) \det(A - tI_n) \det(S) \stackrel{\det S \neq 0}{=} \frac{1}{\det(S)} P_A(t) \det(S) = P_A(t)$

Remark: If $P_A(t) = P_B(t)$ this does not necessarily mean A & B are similar

Example: $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ & $B = I_2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ both have characteristic polynomial $(t-1)^2$

But B is only similar to itself! (Because no matter what non-singular matrix S we pick, we set $S^{-1} \boxed{I_2} S = S^{-1} S = \boxed{I_2}$)

Q: If A & C are similar & λ is an eigenvalue of both, how are the eigenspaces $E_\lambda(A)$ & $E_\lambda(C)$ related?

The next result will help us answer that question for $C = S^{-1}AS$

Proposition 2: If \vec{v} is an eigenvector of C with eigenvalue λ , then $\vec{w} = S\vec{v}$ is an eigenvector of A with eigenvalue λ .

Proof: Use the definition of eigenvectors.

• First $\vec{v} \neq \vec{0}$ & S is singular (that is, $\mathcal{N}(S) = \{\vec{0}\}$), so $\vec{w} = S\vec{v}$ is also $\neq \vec{0}$.

$$\begin{aligned} \bullet \text{ Now, } A\vec{w} &= AS\vec{v} = S \underbrace{(S^{-1}AS)}_{=C} \vec{v} \\ &= S(C\vec{v}) \underset{\substack{\vec{v} \text{ eigenvector} \\ \downarrow}}{=} S\lambda\vec{v} \underset{=C}{=} \lambda(S\vec{v}) = \lambda\vec{w} \end{aligned}$$

Conclude: $A\vec{w} = \lambda\vec{w}$ & $\vec{w} \neq \vec{0}$ so it's an eigenvector of A with eigenvalue λ .

Obs: We can switch the roles of A & C to get the following

Corollary 1: If \vec{w} is an eigenvector of A with eigenvalue λ , then $\vec{v} = S^{-1}\vec{w}$ is an eigenvector of C with eigenvalue λ ,
(writing $(S^{-1})^{-1}C(S^{-1}) = A$)

Combining Prop 2 & Corollary 1, we get an invertible linear transf³
between $E_\lambda(A)$ & $E_\lambda(C)$, namely

$$T: E_\lambda(C) \longrightarrow E_\lambda(A)$$

$$\vec{v} \longmapsto S\vec{v}$$

$$T^{-1}: E_\lambda(A) \longrightarrow E_\lambda(C)$$

$$\vec{w} \longmapsto S^{-1}\vec{w}$$

Consequences ① $\dim E_\lambda(C) = \dim E_\lambda(A)$

② If $\{\vec{v}_1, \dots, \vec{v}_p\}$ is a basis for $E_\lambda(C)$, then $\{S\vec{v}_1, \dots, S\vec{v}_p\}$ is a basis for $E_\lambda(A)$.

Example: $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $C = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ We claim: A & C are similar matrices

$$\bullet P_A(t) = \det \begin{pmatrix} t & -1 \\ 1 & -t \end{pmatrix} = t^2 - 1 = (t+1)(t-1) = P_C(t)$$

$$\bullet E_1(A) = \mathcal{N} \left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) = \mathcal{N} \left(\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \right)$$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \xrightarrow{R_2 \rightarrow R_2 + R_1} \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \xrightarrow{R_1 \rightarrow -R_1} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \rightsquigarrow \begin{matrix} x_1 - x_2 = 0 \\ \boxed{x_1 = x_2} \end{matrix}$$

REF (indep.)

General soln: $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \rightsquigarrow E_1(A) = \text{Sp} \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)$

$$\bullet E_{-1}(A) = \mathcal{N} \left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - (-1) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) = \mathcal{N} \left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right)$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \xrightarrow{R_2 \rightarrow R_2 - R_1} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \rightsquigarrow \begin{matrix} x_1 + x_2 = 0 \\ \boxed{x_1 = -x_2} \end{matrix}$$

REF (indep.)

General soln $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

So $E_{-1}(A) = \text{Sp} \left(\begin{bmatrix} -1 \\ 1 \end{bmatrix} \right)$

Claim: If $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ then if $B = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$

$$\vec{v} \longmapsto A\vec{v}$$

we have $T \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ & $T \left(\begin{bmatrix} -1 \\ 1 \end{bmatrix} \right) = - \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ so $[T]_{BB} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = C$

Then $A = [T]_{\mathbb{E}\mathbb{E}}$ & we can write $C = S^{-1}AS$

with $S = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ (columns = basis B where $[T]_{B_B} = C$)

Let's check this! $S^{-1} \stackrel{\substack{\uparrow \\ \text{det}(S)}}}{=} \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$
2x2 formula

$$S^{-1}AS = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$$

Summary: We showed A & C are similar by explicitly computing the matrix S . For this, we used eigenvectors of A because C was a diagonal matrix. = C ✓
• This example shows how the "diagonalization" process will work.

§ 2 Diagonalization:

Definition: We say an $n \times n$ real matrix A is diagonalizable (over \mathbb{C}) if it is similar to a diagonal matrix $D = \begin{pmatrix} \boxed{d_1} & 0 & \dots & 0 \\ 0 & \dots & \dots & 0 \\ \vdots & \dots & \dots & \vdots \\ 0 & \dots & 0 & \boxed{d_n} \end{pmatrix}$ (with entries in \mathbb{C})
(If we work over \mathbb{C} , the matrix S also has entries in \mathbb{C})

Observe: $P_D(t) = \det(D - tI_n) = \det \begin{pmatrix} d_1 - t & & & 0 \\ & \dots & & \\ & & \dots & \\ 0 & & & d_n - t \end{pmatrix} = (d_1 - t) \dots (d_n - t)$

So the eigenvalues of D are d_1, \dots, d_n (counted with multiplicity, that is, we can have repetitions!)

Since similar matrices have the same characteristic polynomials, we conclude that the entries along the diagonal in D are the eigenvalues of A .

Q: Why is it useful to diagonalize a matrix (whenever possible)?

A: We can perform power operations very fast!

5

Indeed, if $D = S^{-1}AS$, then $D^k = \begin{pmatrix} d_1^k & & 0 \\ & \ddots & \\ 0 & & d_n^k \end{pmatrix} = \begin{pmatrix} d_1^k & & 0 \\ & \ddots & \\ 0 & & d_n^k \end{pmatrix}$

$$D^k = (S^{-1}AS)^k = \underbrace{(S^{-1}AS)(S^{-1}AS)}_{=I_n} \cdots \underbrace{(S^{-1}AS)(S^{-1}AS)}_{=I_n}$$

so the middle terms all cancel out k times & we get

$$D^k = S^{-1}A^k S$$

So $S \boxed{D^k} S^{-1} = \underbrace{S(S^{-1}A^k S)}_{=I_2} S^{-1} = A^k$
 easy to do!

Conclude: $A^k = S D^k S^{-1} = S \begin{pmatrix} d_1^k & & 0 \\ & \ddots & \\ 0 & & d_n^k \end{pmatrix} S^{-1}$ for all $k=1,2,3,\dots$

Our next result characterizes when a matrix A is diagonalizable over \mathbb{C} . As we saw in the example on page 3, we must have a basis for \mathbb{C}^n consisting of eigenvectors of A (over \mathbb{C}^n). We distinguish the complex vs real setting.

Theorem 1 An $n \times n$ matrix A is diagonalizable over \mathbb{C} if and only if A has a set of n linearly independent eigenvectors in \mathbb{C}^n

(2) An $n \times n$ matrix A with real entries is diagonalizable over \mathbb{R} if all eigenvalues of A are real & we have n linearly independent eigenvectors of A in \mathbb{R}^n

Proof: We discuss (1). For statement (2), it suffices to notice that

D will be the same in (1) & (2), so D being real means the eigenvalues of A are all real. Once this is established, we see that to compute $E_\lambda(A)$ for λ real & A with real entries we never leave the world of real (as opposed to complex) entries.

So the basis of eigenvectors in (1) is actually in \mathbb{R}^n . This shows (2) follows from (1).

We prove (1) by showing the two implications

• **One direction:** Assume we have a basis $B = \{v_1, \dots, v_n\}$ for \mathbb{C}^n consisting of eigenvectors of A . Write $S = [v_1, \dots, v_n]$

- S is nonsingular (as a matrix with complex entries) because the columns are li. In particular, S is invertible (again, over \mathbb{C})
- By definition, we know $Av_i = \lambda_i v_i$ for λ_i eigenvalue of A
- We want to show: $S^{-1}AS = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$

$$AS = A[v_1 \dots v_n] = [\lambda_1 v_1 \dots \lambda_n v_n]$$

$$\text{So } S^{-1}AS = S^{-1}[\lambda_1 v_1 \dots \lambda_n v_n] = [\lambda_1 S^{-1}v_1 \dots \lambda_n S^{-1}v_n]$$

$$S^{-1}v_j = S^{-1}(\text{col}_j S) = e_j \left(= \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \text{ at } j^{\text{th}} \text{ spot} \right) \text{ because } S^{-1}S = I_n$$

$$\text{So } S^{-1}AS = [\lambda_1 e_1 \dots \lambda_n e_n] = \begin{bmatrix} \lambda_1 & 0 & & \\ 0 & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} \text{ is diagonal! } \checkmark$$

• **Other direction:** Assume A is diagonalizable & write

$$D = S^{-1}AS \text{ for some invertible matrix } S \text{ \& } D = \text{diagonal}$$

$$\text{Then } SD = AS$$

• Write the column vectors of S as $\{v_1, \dots, v_n\}$ (they are vectors in \mathbb{C}^n)

$$\text{• Write } D = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix}$$

$$\text{Then } SD = [v_1 \dots v_n] \begin{bmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{bmatrix} = [d_1 v_1, \dots, d_n v_n] \quad (*)$$

$$\text{(For example } [v_1 \dots v_n] \begin{bmatrix} d_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = d_1 [v_1 \dots v_n] \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = d_1 v_1 \text{, the others follow by the same reasoning)}$$

$$\text{Now: } AS = [Av_1, \dots, Av_n] \quad (**)$$

Putting (*) & (**) together & comparing column by column gives

$$d_1 \vec{v}_1 = A\vec{v}_1, \quad d_2 \vec{v}_2 = A\vec{v}_2, \quad \dots, \quad d_n \vec{v}_n = A\vec{v}_n \quad (***)$$

Since S is nonsingular $\{\vec{v}_1, \dots, \vec{v}_n\}$ is a basis for \mathbb{C}^n . In particular, all \vec{v}_i 's are $\neq \vec{0}$ & by (***) they are eigenvectors of A . \checkmark

Remark: It is important to notice that the basis of eigenvectors is ORDERED⁷ by the location of the eigenvalues placed along the diagonal of D .

Example 1: $A = \begin{bmatrix} 5 & -6 \\ 3 & -4 \end{bmatrix}$ $P_A(t) = \det \begin{pmatrix} 5-t & -6 \\ 3 & -4-t \end{pmatrix} = (5-t)(-4-t) + 18$
 $= t^2 - t + 2 = (t-2)(t+1)$

Since we have 2 distinct & real eigenvalues we know from Lecture 33 (Consequence 1, page 9), that A has a basis of eigenvectors for \mathbb{R}^2 , so it's diagonalizable over \mathbb{R} .

E_{-1} : Solve $\begin{bmatrix} 6 & -6 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightsquigarrow x_1 - x_2 = 0$ $E_{-1} = \text{Sp} \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)$
 so $x_1 = x_2$

E_2 : Solve $\begin{bmatrix} 3 & -6 \\ 3 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightsquigarrow x_1 - 2x_2 = 0$ $E_2 = \text{Sp} \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} \right)$
 so $x_1 = 2x_2$

$D = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$ & $S = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \rightsquigarrow S^{-1} = \frac{1}{\det S} \begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$
 basis for E_{-1} basis for E_2

Easy Check: $D = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} \stackrel{?}{=} S^{-1}AS = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 5 & -6 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \quad \checkmark$

Example 2: Use this to compute A^{10}

$$A^{15} = S D^{10} S^{-1} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} (-1)^{10} & 0 \\ 0 & 2^{10} \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1024 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 1024 & -1024 \end{bmatrix} = \begin{bmatrix} 2047 & -2046 \\ 1023 & -1022 \end{bmatrix}$$

Example 3 Let's look at Example 1 on page 4 of Lecture 35.

$A = \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}$ eigenvalues = $2+i, 2-i \rightsquigarrow$ so diagonalizable over \mathbb{C} .

$E_{2+i} = \text{Sp} \left(\begin{bmatrix} -1-i \\ 2 \end{bmatrix} \right)$ & $E_{2-i} = \text{Sp} \left(\overline{\begin{bmatrix} -1-i \\ 2 \end{bmatrix}} \right) = \text{Sp} \left(\begin{bmatrix} -1+i \\ 2 \end{bmatrix} \right)$

$$S = \begin{bmatrix} -1-i & -1+i \\ 2 & 2 \end{bmatrix} \quad \Rightarrow \quad D = \begin{bmatrix} 2+i & 0 \\ 0 & 2-i \end{bmatrix}$$

To compute S^{-1} , we use the usual formula for 2×2 matrices (it works for both real & complex matrices!) $\det S = (-1-i)2 - (-1+i)2$

$$S^{-1} = \frac{1}{\det(S)} \begin{bmatrix} 2 & -(-1+i) \\ -2 & -1-i \end{bmatrix} = \frac{1}{-4i} \begin{bmatrix} 2 & 1-i \\ -2 & -1-i \end{bmatrix} = \begin{bmatrix} i/2 & \frac{1+i}{4} \\ -i/2 & \frac{1-i}{4} \end{bmatrix}$$

$$= \frac{(-4i)}{|-4i|^2} = \frac{4i}{16} = \frac{i}{4}$$

Check $D = S^{-1} A S$

$$\begin{bmatrix} 2+i & 0 \\ 0 & 2-i \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} i/2 & \frac{1+i}{4} \\ -i/2 & \frac{1-i}{4} \end{bmatrix} \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} -1-i & -1+i \\ 2 & 2 \end{bmatrix}$$

$$\begin{bmatrix} (-3+2)-i3 & (-3+2)+i3 \\ (2+2)+i2 & (2+2)-i2 \end{bmatrix} = \begin{bmatrix} -1-i3 & -1+i3 \\ 4+i2 & 4-i2 \end{bmatrix}$$

Compare all 4 entries:

$$2+i \stackrel{?}{=} \frac{i}{2}(-1-i3) + (4+i2)\left(\frac{1+i}{4}\right) = \frac{3}{2} - \frac{i}{2} + \frac{1}{2}(2-1+i(2+1))$$

$$0 \stackrel{?}{=} \frac{i}{2}(-1+i3) + (4-i2)\left(\frac{1+i}{4}\right) = -\frac{3}{2} - \frac{i}{2} + \frac{1}{2}(2+1+i(2-1))$$

$$0 \stackrel{?}{=} -\frac{i}{2}(-1-i3) + (4+i2)\left(\frac{1-i}{4}\right) = -\frac{3}{2} + \frac{i}{2} + \frac{1}{2}(2+1+i(-2+1))$$

$$2-i \stackrel{?}{=} -\frac{i}{2}(-1+i3) + (4-i2)\left(\frac{1-i}{4}\right) = \frac{3}{2} + \frac{i}{2} + \frac{1}{2}(2-1+i(-2-1))$$

Q: What about other sizes?

In Lecture 33 Thm page 7, we saw that eigenvectors with different eigenvalues are li. This has a powerful consequence:

Theorem Fix $B_\lambda = \{\vec{v}_1, \dots, \vec{v}_p\}$ a basis for the eigenspace E_λ of a matrix A
 $B_\mu = \{\vec{w}_1, \dots, \vec{w}_q\}$ E_μ

Then $B_\lambda \cup B_\mu = \{\vec{v}_1, \dots, \vec{v}_p, \vec{w}_1, \dots, \vec{w}_q\}$ is linearly indep.

Proof: Write $\underbrace{(\alpha_1 \vec{v}_1 + \dots + \alpha_p \vec{v}_p)}_{\text{it's an eigenvector in } E_\lambda} + \underbrace{(\beta_1 \vec{w}_1 + \dots + \beta_q \vec{w}_q)}_{\text{it's an eigenvector in } E_\mu} = \vec{0}$

$$\vec{v} = \alpha_1 \vec{v}_1 + \dots + \alpha_p \vec{v}_p = -(\beta_1 \vec{w}_1 + \dots + \beta_q \vec{w}_q) = -\vec{w}$$

$$\left. \begin{array}{l} \text{So } A\vec{v} = \lambda \vec{v} \\ \text{"} \\ A(-\vec{w}) = \mu(-\vec{w}) = \mu \vec{v} \end{array} \right\} \begin{array}{l} (\lambda - \mu) \vec{v} = \vec{0} \text{ \& } \lambda \neq \mu \\ \text{for } \vec{v} = \vec{0} \\ \vec{w} = \vec{0} \end{array}$$

But then $\vec{0} = \alpha_1 \vec{v}_1 + \dots + \alpha_p \vec{v}_p$ & li so $\alpha_1 = \dots = \alpha_p = 0$
 $\vec{0} = \beta_1 \vec{v}_1 + \dots + \beta_q \vec{v}_q$ ————— $\beta_1 = \dots = \beta_q = 0$

Conclude $\{\vec{v}_1, \dots, \vec{v}_p, \vec{v}_{p+1}, \dots, \vec{v}_q\}$ are li. \square

Obs: The same idea works if we have more than 2 eigenspaces

Theorem: A $n \times n$ with eigenvalues $\lambda_1, \dots, \lambda_s$ (counted with multiplicity)

Then: A is diagonalizable over \mathbb{C} if and only if

$$\dim E_{\lambda_1} + \dots + \dim E_{\lambda_s} = n$$

(This will ensure that we have a basis of eigenvectors = $B_{\lambda_1} \cup B_{\lambda_2} \cup \dots \cup B_{\lambda_s}$
 (just collect the basis for each E_{λ_i})

Example 4 $A = \begin{bmatrix} 25 & -8 & 30 \\ 24 & -7 & 30 \\ -12 & 4 & -14 \end{bmatrix}$

$$\begin{aligned} \rightsquigarrow P_A(t) &= -t^3 + 4t^2 - 5t + 2 \\ &= (1-t)^2(2-t) \end{aligned}$$

We compute the 2 eigenspaces:

$$\left. \begin{array}{l} E_1 = \mathcal{N} \left(\begin{bmatrix} 24 & -8 & 30 \\ 24 & -8 & 30 \\ -12 & 4 & -14 \end{bmatrix} \right) = \text{Sp} \left(\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 3 \\ 4 \end{bmatrix} \right) \rightsquigarrow \dim = 2 \\ E_2 = \mathcal{N} \left(\begin{bmatrix} 23 & -8 & 30 \\ 24 & -9 & 30 \\ -12 & 4 & -16 \end{bmatrix} \right) = \text{Sp} \left(\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \right) \rightsquigarrow \dim = 1 \end{array} \right\} \begin{array}{l} 2+1=3 \text{ so} \\ A \text{ is diagonalizable} \end{array}$$

In fact, we have $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ & $S = \begin{bmatrix} 1 & -4 & 2 \\ 3 & 3 & 2 \\ 0 & 4 & -1 \end{bmatrix}$ 10

§3 Real Symmetric Matrices:

In addition to having only real eigenvalues, real symmetric matrices are always diagonalizable. Furthermore, we can choose an orthonormal basis of eigenvectors. Here's the statement, followed by an example:

Theorem Fix a real $n \times n$ symmetric matrix A . Then

(1) We can find a basis $\{\vec{v}_1, \dots, \vec{v}_n\}$ of eigenvectors for \mathbb{R}^n that is orthonormal, so $\vec{v}_i \cdot \vec{v}_j = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{otherwise} \end{cases}$

(2) If $Q = [v_1, \dots, v_n]$, then $Q^{-1} = Q^T$ & $D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} = Q^T A Q$

Example: $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

$P_A(t) = (1-t)^2 - 1 = t(t-2)$

$E_0 = \text{Sp} \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)$

$B = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ is orthogonal, but norms are not 1, so we need to adjust for this.

$E_2 = \text{Sp} \left(\begin{bmatrix} -1 \\ 1 \end{bmatrix} \right)$

$\| \begin{bmatrix} 1 \\ 1 \end{bmatrix} \| = \sqrt{2}$ & $\| \begin{bmatrix} -1 \\ 1 \end{bmatrix} \| = \sqrt{2}$

$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

Check $Q^T Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ as expected!

$D = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$