SOLUTIONS

Midterm 1
Math 2568 - Linear Algebra (Section 30)

Prof. Cueto

Friday Jan. 31st 2020

- The use of class notes, book, formulae sheet or calculator is **not permitted**.
- In order to get full credit, you **must**:
  a) get the **correct answer**, and
  b) **show all your work** and/or explain the reasoning that leads to that answer.
- Answer the questions **in the spaces provided** on the question sheets. If you run out of room for an answer, continue on the back of the page.
- Please make sure the solutions you hand in are **legible and lucid**.
- You have **fifty-five minutes** to complete the exam.
- Do not forget to write your full name (in PRINT) in the space provided below and on the bottom of the last page.

Full Name (Print): ______________________________

Good luck!

Exercise 1. **[8 points]** Consider the homogeneous system of linear equations (with parameter a):

$$\begin{cases} x_1 + 2x_2 + x_3 = 0, \\ -x_1 - x_2 + x_3 = 0, \\ 3x_1 + 4x_2 + a\,x_3 = 0. \end{cases}$$

a) For what values of a does the system have *a unique solution*? Write the explicit solution in vector form for each value of a.

We find the REF of the system

$$\begin{bmatrix} 1 & 2 & 1 \\ -1 & -1 & 1 \\ 3 & 4 & a \end{bmatrix} \xrightarrow{R_2 \to R_2 + R_1} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 3 & 4 & a \end{bmatrix} \xrightarrow{R_3 \to R_3 - 3R_1} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & -2 & a-3 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 + 2R_2} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & a+1 \end{bmatrix}$$

The system has a unique solution only when $a + 1 \neq 0$, so $\boxed{a \neq -1}$

In all such cases the only solution must be the trivial one $\underline{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

b) For what values of a do we have *infinitely many solutions*? For each of them, write the general solution in vector form.

We use the calculation from item (a). Answer is $\boxed{a = -1}$.

We have 1 independent variable (x_3) & 2 dependent ones

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 \to R_1 - 2R_2} \underbrace{\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}}_{\text{RET}}$$

$$\begin{cases} x_1 = 3x_2 \\ x_2 = -2x_2 \\ x_3 \text{ any} \end{cases}$$

Answer: $\underline{x} = x_2 \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$

Exercise 2. [10 points] Consider the vectors $\mathbf{v_1} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$, $\mathbf{v_2} = \begin{bmatrix} -2 \\ 0 \\ 5 \end{bmatrix}$ and $\mathbf{v_3} = \begin{bmatrix} 0 \\ 6 \\ 5 \end{bmatrix}$.

a) Verify that $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ is linearly dependent by writing the zero vector $\mathbf{0}$ in $\mathbb{R}^3$ as a linear combination of these three vectors.

We propose the system $\begin{bmatrix} 1 & -2 & 0 \\ 3 & 0 & 6 \\ 0 & 5 & 5 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 1 & -2 & 0 \\ 3 & 0 & 6 \\ 0 & 5 & 5 \end{bmatrix} \xrightarrow[R_2 \to R_2 - 3R_1]{} \begin{bmatrix} 1 & -2 & 0 \\ 0 & 6 & 6 \\ 0 & 5 & 5 \end{bmatrix} \xrightarrow[\substack{R_2 \to \frac{1}{6}R_2 \\ R_3 \to R_3 - R_2}]{} \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow[R_1 \to R_1 + 2R_2]{} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

so $\begin{cases} a_1 = -2a_3 \\ a_2 = -a_3 \\ a_3 \text{ any} \end{cases}$ set $a_3 = 1$, for example

$$\boxed{\vec{0} = -2\vec{v}_1 - \vec{v}_2 + \vec{v}_3}$$

b) Express $\mathbf{v_3}$ as a linear combination of $\mathbf{v_1}$ and $\mathbf{v_2}$.

$\vec{v}_3 = 2\vec{v}_1 + \vec{v}_2$

c) Show that $\{\mathbf{v_1} + \mathbf{v_2} + \mathbf{v_3}, 2\mathbf{v_2}, \mathbf{v_3}\}$ is also linearly dependent.

Say $k_1(v_1 + v_2 + v_3) + k_2(2v_2) + k_3 v_3 = \vec{0}$

Reorder: $k_1 v_1 + (k_1 + 2k_2) v_2 + (k_1 + k_3) v_3 = \vec{0}$

Now we know $\{v_1, v_2, v_3\}$ is ld, eg. for $k_1 = -2$, $k_1 + 2k_2 = -1$, $k_1 + k_3 = 1$ works.

So $k_1 = -2$, $k_2 = \frac{1}{2}(-1+2) = \frac{1}{2}$, $k_3 = 1 - (-2) = 3$

We found a nontrivial linear combination!

Exercise 3. **[10 points]** Consider the matrix $A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 2 \\ 1 & 1 & 4 \end{bmatrix}$.

a) Show that A is invertible by computing the inverse matrix A^{-1}.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 2 & 1 & 2 & 0 & 1 & 0 \\ 1 & 1 & 4 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\substack{R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1}]{} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 4 & -2 & 1 & 0 \\ 0 & 1 & 5 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 - R_2]{} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 4 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right] \xrightarrow[R_1 \to R_1 + R_3]{} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & 1 \\ 0 & 1 & 4 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right]$$

$$\xrightarrow[R_2 \to R_2 - 4R_3]{} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & 1 \\ 0 & 1 & 0 & -6 & 5 & -4 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right] \qquad \underbrace{}_{A^{-1}}$$

Answer: $\boxed{A^{-1} = \begin{bmatrix} 2 & -1 & 1 \\ -6 & 5 & -4 \\ 1 & -1 & 1 \end{bmatrix}}$

b) Write the vector form of the general solution of $A\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ and determine $\|\mathbf{x}\|$.

(*Hint:* Use item a).)

$$x = A^{-1}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 1 \\ -6 & 5 & -4 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2-1-1 \\ -6+5+4 \\ 1-1-1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix}$$

Answer: $\|x\| = \sqrt{3^2 + (-1)^2} = \boxed{\sqrt{10}}$

c) Are the columns of A linearly independent? Justify your answer.

YES $\underbrace{\begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 2 \\ 1 & 1 & 4 \end{bmatrix}}_{=A} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ has a unique solution because A is invertible.

Exercise 4. **[10 points]**

a) Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$. Calculate ABA, BAB and $-2((AB)^3)^T$.

$$AB = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$$

Assoc gives $ABA = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}$

$BAB = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}$

$(AB)^3 = (ABA)(BAB) \underset{\text{Assoc.}}{=} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

So $-2(AB^3)^T = -2\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \boxed{\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}}$

b) Assume P, Q and R are invertible $n \times n$ matrices with $PQR = I_n$. Express Q^{-1} in terms of P and R.

$PQR = I_n$ mult by P^{-1} to the left & R^{-1} to the right.

$P^{-1}PQRR^{-1} = P^{-1}I_nR^{-1}$

$\boxed{Q = P^{-1}R^{-1}}$

$\boxed{Q^{-1} = RP}$

c) Let P, Q be two $n \times n$ matrix and write $R = PQ$. Show that if Q is singular, so is R.

If Q is singular, we can find $\underline{x} \neq 0$ in $\mathbb{R}^n$ with $Q\underline{x} = 0$.

So $R\underline{x} = (PQ)\underline{x} \underset{\text{Assoc}}{=} P(Q\underline{x}) = P \cdot 0 = 0$. & $\underline{x} \neq 0$.

Conclude: R is $\underline{\text{singular}}$.

Exercise 5. [12 points]

Determine whether each of the statements is True or False. **No justification is necessary.**

a) *True* *False* A homogeneous system of equations can be inconsistent.

b) *True* *False* A linear system with less equations than unknowns cannot have a unique solution.

c) *True* *False* Two linear systems are equivalent if they have the same solution set.

d) *True* *False* If a square matrix A is row equivalent to the identity matrix, then A is invertible.

e) *True* *False* Applying elementary row operations on an augmented matrix can change the solutions set of the associated linear system.

f) *True* *False* A consistent 3×2 system of linear equations can have exactly two solutions.

For Grader's use only:

1		2			3			4			5						TOTAL
a	b	a	b	c	a	b	c	a	b	c	a	b	c	d	e	f	