SOLUTIONS

Midterm 2
Math 2568 - Linear Algebra (Section 30)

Prof. Cueto

Friday March 6th 2020

- The use of class notes, book, formulae sheet or calculator is **not permitted**.
- In order to get full credit, you **must**:
  a) get the **correct answer**, and
  b) **show all your work** and/or explain the reasoning that leads to that answer.
- Answer the questions **in the spaces provided** on the question sheets. If you run out of room for an answer, continue on the back of the page.
- Please make sure the solutions you hand in are **legible and lucid**.
- You have **fifty-five minutes** to complete the exam.
- The total number of points for this midterm is **fifty**.
- Do not forget to write your full name (in PRINT) in the space provided below and on the bottom of the last page.

Full Name (Print): ______________________________

Good luck and have a fun Spring Break!

Exercise 1. [**12 points**] Consider the matrix $A = \begin{bmatrix} 1 & -1 & 2 & 3 \\ 2 & -2 & 5 & 4 \\ 1 & -1 & 0 & -7 \end{bmatrix}$.

a) Find a **basis** for the Null Space of A.

$$\begin{bmatrix} 1 & -1 & 2 & 3 \\ 2 & -2 & 5 & 4 \\ 1 & -1 & 0 & -7 \end{bmatrix} \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & -2 & -10 \end{bmatrix} \xrightarrow[R_3 \to R_3 + 2R_2]{} \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & -30 \end{bmatrix}$$

$$\xrightarrow[R_3 \to -\frac{R_3}{30}]{} \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[\substack{R_2 \to R_2 + 2R_3 \\ R_1 \to R_1 - 3R_3}]{} \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_1 \to R_1 - 2R_2]{} \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

indep

so $x_3 = x_4 = 0$

$x_1 = x_2$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_2 \\ 0 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad \text{Basis } \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

b) Compute the **dimension** of the column space of A (that is, the Range of A).

Use $\dim \mathcal{N}(A) + \dim \text{Range}(A) = \#\text{cols}(A)$

$1 + \dim \text{Range}(A) = 4$

So $\dim \text{Range}(A) = \boxed{3}$

(*) Alt way: Look at (a) for $A' = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $\dim \text{row span}(A) = \dim \text{row span}(A')$ clearly 3.

c) Show that the **rows** of A are linearly independent vectors.

Know $\text{rank}(A^T) = \dim \text{Row Space}(A)$

$\text{rank}(A^T) = \text{rank}(A) = 3$

We have 3 rows & so they must be linearly independent

(*)

Exercise 2. [**8 points**] Fix $\mathbf{w_1} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix}$, $\mathbf{w_2} = \begin{bmatrix} 0 \\ 1 \\ 3 \\ 1 \end{bmatrix}$ and $\mathbf{w_3} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$.

a) Using Gram-Schmidt find an **orthogonal basis** for the span of $\{\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}\}$.

$u_1 = w_1 \qquad \|u_1\|^2 = 1+4+1 = 6 \qquad u_1 \cdot w_2 = 1+6+1 = 8$

$u_1 w_3 = 1+2 = 3$

$$u_2 = w_2 - \frac{u_1 w_2}{\|u_1\|^2} u_1 = \begin{bmatrix} 0 \\ 1 \\ 3 \\ 1 \end{bmatrix} - \frac{8}{6}\begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 3 \\ 1 \end{bmatrix} - \frac{4}{3}\begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1/3 \\ 1/3 \\ -1/3 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

$\|u_2\|^2 = \frac{1}{9}\, 3 = \frac{1}{3} \qquad u_2 \cdot w_3 = \frac{1}{3}\begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} = 0$

$$u_3 = w_3 - \frac{u_1 w_3}{\|u_1\|^2} u_1 - \overset{=0}{\frac{u_2 . w_3}{\|u_2\|^2}} u_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} - \frac{3}{6}\begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1-1/2 \\ 1-1 \\ -1/2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1/2 \\ 0 \\ -1/2 \end{bmatrix}$$

$$\mathcal{B} = \left\{ \underset{u_1}{\begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix}}, \underset{u_2}{\frac{1}{3}\begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \end{bmatrix}}, \underset{u_3}{\frac{1}{2}\begin{bmatrix} 2 \\ 1 \\ 0 \\ -1 \end{bmatrix}} \right\}$$

b) Give the **orthonormal basis** associated to the result from item a).

Need to ~~divide~~ multiply v by $\frac{1}{\|v\|}$.

$v = u_1 \qquad \|u_1\| = \sqrt{6}$

$v = u_2 = \frac{1}{3}\|u_2\| = \frac{1}{3}\frac{1}{\sqrt{3}}$

$v = u_3 = \frac{1}{2}\left\|\begin{bmatrix} 2 \\ 1 \\ 0 \\ -1 \end{bmatrix}\right\| = \frac{1}{2}\sqrt{4+2} = \frac{\sqrt{6}}{2}$

$$\mathcal{B}' = \left\{ \frac{1}{\sqrt{6}}\begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{3}}\begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \frac{1}{\sqrt{6}}\begin{bmatrix} 2 \\ 1 \\ 0 \\ -1 \end{bmatrix} \right\}$$

Exercise 3. [**8 points**] Consider the space $\mathcal{P}_2$ of polynomials $P(x)$ of degree at most 2. Show that $B = \{1 + x + x^2, -1 + x, 1 + 2x + x^2\}$ is a basis of $\mathcal{P}_2$.

Use coordinates with respect to the standard basis $B' = \{1, x, x^2\}$

$$[1+x+x^2]_{B'} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad [-1+x]_{B'} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \quad [1+2x+x^2]_{B'} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

Check li (use rows!)

$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \xrightarrow[\substack{R_2 \to R_2 + R_1 \\ R_3 \to R_3 - R_1}]{} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow[R_2 \leftrightarrow R_3]{} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 2R_2]{} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- We conclude 3 "vectors" are li
- Since $\dim \mathcal{P}_2 = 3$, they automatically span & so B is a basis for $\mathcal{P}_2$.

Exercise 4. [**10 points**] Fix the planes $\pi_1: x+5y+z=3$ and $\pi_2: 2x-y+3z=1$.

a) Show that the two planes are perpendicular to each other, i.e., their normal vectors are perpendicular.

$$\begin{bmatrix} 1 & 5 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = 2-5+3=0.$$

b) Find the line where the two planes intersect.

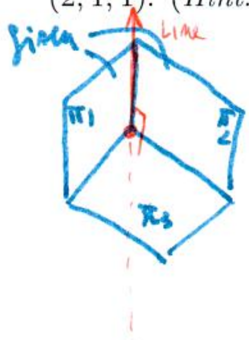
$$\begin{cases} x+5y+z=3 \\ 2x-y+3z=1 \end{cases} \qquad \left[\begin{array}{ccc|c} 1 & 5 & 1 & 3 \\ 2 & -1 & 3 & 1 \end{array}\right] \xrightarrow[R_2 \to R_2 - 2R_1]{} \left[\begin{array}{ccc|c} 1 & 5 & 1 & 3 \\ 0 & -11 & 1 & -7 \end{array}\right]$$

$$\xrightarrow[R_2 \to -\frac{R_2}{11}]{} \left[\begin{array}{ccc|c} 1 & 5 & 1 & 3 \\ 0 & 1 & -\frac{1}{11} & \frac{7}{11} \end{array}\right] \xrightarrow[R_1 \to R_1 - 5R_2]{} \left[\begin{array}{ccc|c} 1 & 0 & \frac{16}{11} & -\frac{2}{11} \\ 0 & 1 & -\frac{1}{11} & \frac{7}{11} \end{array}\right]$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\frac{2}{11} - \frac{16}{11}x_3 \\ \frac{7}{11} + \frac{1}{11}x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\frac{2}{11} \\ \frac{7}{11} \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -\frac{16}{11} \\ \frac{1}{11} \\ 1 \end{bmatrix}$$

Line passes through $(-\frac{2}{11}, \frac{7}{11}, 0)$ & has direction $\begin{bmatrix} -\frac{16}{11} \\ \frac{1}{11} \\ 1 \end{bmatrix}$

c) Find a plane π_3 that is perpendicular to both π_1 an π_2, passing through the point $(2,1,1)$. (*Hint:* Look at one of the corners in this room to visualize this.)



Point $= (2, 1, 1)$
normal $=$ direction of the line

Equation:

$$-\frac{16}{11}(x-2) + \frac{1}{11}(y-1) + 1(z-1) = 0$$

$$-16(x-2) + (y-1) + 11(z-1) = 0$$

$$\boxed{-16x + y + 11z = -20.}$$

Exercise 5. [12 points]

Determine whether each of the statements is True or False. **No justification is necessary.**

a) *True* *False* The space of 2×3 matrices has dimension 7.

b) *True* *False* If 2 matrices A, B of size $2{\times}2$ are linear indep., so are A and $A{+}B$.

c) *True* *False* A set of vectors spanning $\mathbb{R}^4$ has at least 4 vectors.

d) *True* *False* For any two vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^2$ we have $\|\vec{u} - 2\vec{v}\| = \|\vec{u}\| + 2\|\vec{v}\|$.

e) *True* *False* The set $\{f \text{ in } \mathrm{C}[0,1] \text{ with } \int_0^1 f(x)\,dx = 1\}$ is a subspace of $\mathrm{C}[0,1]$, where $\mathrm{C}[0,1]$ is the set of continuous real-valued functions on $[0,1]$.

f) *True* *False* The set of functions $\{1, e^x, \sin(x)\}$ is a basis for $\mathrm{C}[0,1]$.

For Grader's use only:

1			2		3	4			5						TOTAL
a	b	c	a	b		a	b	c	a	b	c	d	e	f	