Lecture VI: Right cosets and Normal subgroups

Recall: G group, $H \leq G$ subgroup

We defined an equivalence relation on G: $x \sim_L y \iff x^{-1}y \in H$

G/H = set of equiv classes modulo $\sim_L$. Name: Left cosets

$= \{xH : x \in G\}$

§6.1 Right cosets:

We can define $H\backslash G :=$ the set of right cosets modulo H in a similar way.

Definition: $x \underset{R}{\sim} y$ if and only if $yx^{-1} \in H$.

- It's easy to check $\underset{R}{\sim}$ is an equivalence relation when $H \leq G$.

$\Rightarrow H\backslash G = G/_{\underset{R}{\sim}}$ = equivalence classes modulo $\underset{R}{\sim}$.

Its elements are of the form Hx for $x \in G$

Clearly: $Hx = Hy \iff yx^{-1} \in H$

As before, $|G| = |H\backslash G| \cdot |H|$.

In particular, $|H\backslash G| = |G/H| = (G:H)$ if G is finite

Proposition: $\Psi : G/H \longrightarrow H\backslash G$ is a bijection

$xH \longmapsto Hx^{-1}$

Proof:
- Ψ is well-defined: $xH = yH \iff x^{-1}y \in H$.

$Hx^{-1} = Hy^{-1} \iff y^{-1}(x^{-1})^{-1} = y^{-1}x \in H$

But $x^{-1}y \in H \Rightarrow (x^{-1}y)^{-1} = y^{-1}x \in H$ because $H \leq G$.

- Ψ is one-to-one: Pick x, y with $Hx^{-1} = Hy^{-1}$. Then, $y^{-1}(x^{-1})^{-1} = y^{-1}x \in H$

So $(y^{-1}x)^{-1} = x^{-1}y \in H$. By definition, $x \sim_L y$ so $xH = yH$.

- Ψ is onto: $Hy = \Psi(y^{-1}H)$ $\forall y \in G$. $\square$

⚠ We have $G = \coprod_{\alpha\in A} g_\alpha H = \coprod_{\alpha \in A} H g'_\alpha$. But this does NOT mean that $g_\alpha H = H g'_{\alpha'}$. If so $g_\alpha \in H g'_{\alpha'}$ so we should have $g_\alpha H = H g_\alpha$.

Only special subgroups H will allow for such identifications, namely for normal subgroups.

§6.2 Normal subgroups:

__Definition:__ Fix a group G and $H \leq G$ a subgroup. We H is a __normal subgroup__ and write $H \trianglelefteq G$ if $\forall\, x \in G$ & $h \in H$ we have $x h x^{-1} \in H$.

Equivalently, $H \trianglelefteq G$ if for every $x \in G$ we have $xH = Hx$ as subsets of G.

In short: left cosets are the same as right cosets.

__Main point:__ it is only when $H \trianglelefteq G$ that G/H has a natural group structure inherited from G. We will call G/H the __quotient group__.

__Examples:__ (1) If G is abelian, then every subgroup is normal because
$x h x^{-1} = x x^{-1} h = h \in H$ if $x \in G$ & $h \in H$.

(2) $G = \mathrm{Free}(\{a,b\})$ and $H = \langle x y x^{-1} y^{-1} : x, y \in G \rangle$
$z \in G$ & h in H, then $z h z^{-1} h \in H$ (it's one of the generators). This gives $z h z^{-1} \in H h = H$, so $H \trianglelefteq G$.

§6.3 Group structure on G/H:

__Recall:__ An element of the set G/H is a subset of G. It has the form
$gH := \{ g h \mid h \in H \}$ for some (NOT uniquely determined) $g \in G$.

Q: How can we "multiply" two such sets and get another such set?

Guess: $(g_1 H) * (g_2 H) = (g_1 g_2) H$

Issue: Since the definition involves choosing g_1, g_2 it may very well NOT be well defined. Here is the dilemma:

Q: If $g_1 \sim_L g_1'$ & $g_2 \sim_L g_2'$, is it true that $g_1 g_2 \sim_L g_1' g_2'$?

A: NOT always!

For example $G = S_4 \geq H = S_3 = \{ \sigma \in S_4 \mid \sigma(4) = 4 \}$

$g_1 = (14) \sim_L (124) = (14)(12) = g_1'$ because $g_1^{-1} g_1' = (14)(14)(12) = (12) \in H$

$g_2 = (24) \sim_L (234) = (24)(23) = g_2'$ $\quad g_2^{-1} g_2' = (24)(24)(23) = (23) \in H$

but $g_1 g_2 = (14)(24) = (142) \notin H$

$g_1' g_2' = (14)(12)(24)(23) = (123) \in H$ so $g_1 g_2 \not\sim_L g_1' g_2'$. □

Everything is ok if H is normal in G!

Theorem: If $H \trianglelefteq G$, then G/H is a group under $g_1 H * g_2 H = g_1 g_2 H$.

Proof: • We check $*$ is well-defined if $H \trianglelefteq G$.

Assume $g_1 \sim_L g_1'$ (ie $h_1 := g_1^{-1} g_1' \in H$)

$g_2 \sim_L g_2'$ (ie $h_2 := g_2^{-1} g_2' \in H$)

Then, $(g_1 g_2)^{-1} (g_1' g_2') =$

$$= g_2^{-1} \underbrace{(g_1^{-1} g_1')}_{= h_1 \in H} g_2' = g_2^{-1} h_1 g_2' = \underbrace{(g_2^{-1} h_1 g_2)}_{\substack{\in H \\ \text{because } H \trianglelefteq G}} \underbrace{g_2^{-1} g_2'}_{h_2 \in H}$$

lies in H, so $g_1 g_2 \sim_L g_1' g_2'$

Thus $g_1 H * g_2 H := g_1 g_2 H$ is well-defined

• Neutral Element: eH $(= H)$

• Next, we need to check $*$ & eH satisfies the 3 properties determining a group

(1) Associativity: $g_1 H * (g_2 H * g_3 H) = g_1 H * (g_2 g_3 H) = g_1 (g_2 g_3) H$

$$\underset{\text{Assoc in } G}{=} ((g_1 g_2) g_3) H = (g_1 g_2) H * g_3 H = (g_1 H * g_2 H) * g_3 H.$$

(2) eH is Neutral Element: $gH * eH = (ge) H = gH = (eg) H = eH * gH$

$ge = eg = g$
(e Neutral in G)

(3) Inverses: $gH * g^{-1}H = (gg^{-1})H = eH = (g^{-1}g)H = g^{-1}H * gH$, so $(gH)^{-1} = g^{-1}H$ by definition.

§6.4 Examples:

In general, it is hard to build normal subgroups, and we will see some tricks in the future. For now, we'll construct a bunch of examples.

Example 1: $G = S_4 \geq H = \{e, \underbrace{(12)(34), (13)(24), (14)(23)}_{\text{all elements of cycle type } 2+2}\}$

Check: H is a subgroup

- $e \in H$
- Closed under inverses ✓ $\sigma^2 = e$ for all $\sigma \in H$ since $\mathrm{ord}(\sigma) = 1$ or 2 $\forall \sigma \in H$
- ________ multiplication:

$$(12)(34)\,(13)(24) = (14)(23)$$
$$(12)(34)\,(14)(23) = ((12)(34))^2(13)(24) = (13)(24)$$
$$(13)(24)\,(14)(23) = (12)(34)$$
$$(13)(24)\,(12)(34) = (14)(23)$$
$$(14)(23)\,(12)(34) = (13)(24)$$
$$(14)(24)\,(13)(24) = (12)(34)$$

Check: H is normal.

This is a consequence of the following result:

Lemma: Given $\tau, \sigma \in S_n$ $\tau\sigma\tau^{-1}$ has the same cycle type as σ

Proof: By induction on number of cycles featured in σ

- **Base case:** $\sigma = (x_1, \dots x_\ell)$ an ℓ-cycle

  Then $\tau\sigma\tau^{-1} = (\tau(x_1)\,\tau(x_2) \cdots \tau(x_\ell))$ is also an ℓ-cycle.

- **Inductive step:** $\tau(\sigma_1\sigma_2 \cdots \sigma_r)\tau^{-1} = (\tau\sigma_1\tau^{-1})(\tau\sigma_2\tau^{-1}) \cdots (\tau\sigma_r\tau^{-1})$

is again a product of disjoint cycles of the same length as $\sigma = \sigma_1 \sigma_2 \cdots \sigma_r$ $\square$

$H = \{e, (12)(34), (13)(24), (14)(23)\}$ is a subgroup of S_5 but it is <u>not</u> normal because $\tau = (15) \in S_5$ $\quad \sigma = (12)(34) \in H$ but

$$\tau\sigma\tau^{-1} = (15)(12)(34)(15) = (15)(12)(15)(15)(34)(15) = (\tau(1)\,\tau(2))(\tau(3)\,\tau(4))$$
$$= (5\,2)(3\,4) \notin H.$$