Lecture VII: Normal subgroups, Presentations & Group homomorphisms

§7.1 Normal subgroups:

Recall: G group, $H \leq G$ subgroup is <u>normal</u> if $xH = Hx \quad \forall x \in G$

Equivalently, $xHx^{-1} = H \quad \forall x \in G$.

Example 1: (§6.4) $S_4 \trianglerighteq H = \{e, (12)(34), (13)(24), (14)(23)\}$

Lemma: If $H \trianglelefteq S_n$ & $\sigma \in H$, then all the permutations with the same cycle type as σ must lie in H.

Example 2: $G = D_{2n} = \langle s, r \mid s^2 = r^n = e, \ srs = r^{-1} \rangle$

$$\begin{array}{l} \ \ \ \rotatebox{90}{$\leq$} \\ H = \langle r \rangle = \{e, r, r^2, \ldots, r^{n-1}\} \end{array}$$

Claim: H is normal

Proof: By direct calculation. A typical element of G has the form $x = s^a r^b \quad a = 0, 1, \ b = 0, 1, \ldots, n-1. \quad \& \quad x^{-1} = r^{-b} s^a$

Any $h \in H$ is of the form $h = r^l$ for $l = 0, \ldots, n-1$

$$\Rightarrow xhx^{-1} = s^a r^{-b} r^l r^b s^a = s^a r^l s^a = \begin{cases} r^l & \text{if } a = 0 \\ sr^l s = r^{-l} & \text{if } a = 1 \end{cases}$$

In both cases, we get $xhx^{-1} \in H$.

Example 3: $G = S_3 \trianglerighteq N = \langle (123) \rangle = \{e, (123), (132)\}$ (Use N when the subgroup is normal)

Example 4: $G = GL_2(\mathbb{R}) = 2\times 2$ matrices with $\det \neq 0$.

$$\begin{array}{l} \rotatebox{90}{$\leq$} \\ N = \{x \in GL_2(\mathbb{R}) : \det(X) = 1\} =: SL_2(\mathbb{R}). \end{array}$$

- Since $\det(AB) = \det(A)\det(B) \Rightarrow N$ is closed under mult.
- $1^{-1} = 1$ & $\det(I_2) = 1$, so N is closed under inverses & $I_2 \in N$

$\Rightarrow N \leq G$.

- Since $\det(TAT^{-1}) = \det(A)$, we see $N \trianglelefteq G$.

Remark [HW2] $\exists$ G non-abelian where <u>all</u> its subgroups are normal.

Lemma 1: Let G be a group and $H \leq G$ a subgroup. Assume $H = \langle A \rangle$ for some A. We have $H \trianglelefteq G$ if, and only if $xhx^{-1} \in H$ $\forall x \in G$ $\forall h \in A$.

Proof: ($\Rightarrow$) True because $A \subseteq H$

($\Leftarrow$) **Note:** (1) $a h_{i_1} \cdots h_{i_r} a^{-1} = \underbrace{a h_{i_1} a^{-1}}_{\in H} \underbrace{a h_{i_2} a^{-1}}_{\in H} \cdots \underbrace{a h_{i_r} a^{-1}}_{\in H} \in H$ if $h_{i_1}, \dots h_{i_r} \in A$.

and (2) $a h_i^{-1} a^{-1} = (a h_i a^{-1})^{-1} \in H$ if $h \in A$.

These 2 conditions imply it's enough to check the condition for the generators of H, not for all elements in H, because $a h_{i_1}^{\ell_{i_1}} \cdots h_{i_r}^{\ell_{i_r}} a^{-1} = (a h_{i_1} a^{-1})^{\ell_{i_1}} \cdots (a h_{i_r} a^{-1})^{\ell_{i_r}}$. $\square$

Lemma 2: Let G be a group and $H \leq G$ a subgroup. Assume $G = \langle B \rangle$ for some B.

Then, we have $H \trianglelefteq G$ if, and only if, $b^{\varepsilon} h b^{-\varepsilon} \in H$ $\forall h \in H$ $\forall b \in B$ for $\varepsilon = \pm 1$

Proof: Write any x in G as $x = b_1^{\varepsilon_1} \cdots b_\ell^{\varepsilon_\ell}$ with $\varepsilon_i = \pm 1$ $b_1, \dots, b_\ell \in B$

Then, $b_1^{\varepsilon_1} \cdots b_\ell^{\varepsilon_\ell} h (b_1^{\varepsilon_1} \cdots b_\ell^{\varepsilon_\ell})^{-1} = b_1^{\varepsilon_1} \cdots \underbrace{(b_\ell^{\varepsilon_\ell} h b_\ell^{-\varepsilon_\ell})}_{\in H} \cdots b_1^{-\varepsilon_1} \in H$ by induction on ℓ.

§7.2 Presentation of a group:

Equipped with the notion of a normal subgroups, we can define presentations of groups in a formal way.

Definition: Fix G group and $X \subseteq G$ subset, we define $\mathcal{N}_X$ as the smallest normal subgroup of G containing X

Lemma: $$\mathcal{N}_X = \bigcap_{\substack{N \trianglelefteq G \\ X \subseteq N}} N$$

Proof:
- $G \trianglelefteq G$, so the intersection is non-empty
- intersections of normal subgroups is again a normal subgroup

$\Big\}$ (RHS) $\trianglelefteq G$

- $X \subseteq$ (RHS) & (RHS) $\subseteq \mathcal{N}_X$
- By construction, if $N \trianglelefteq G$ & $X \subseteq N$, then (RHS) $\subseteq N$.

Definition: Fix A a set and $\mathcal{R} \subseteq \mathrm{Free}(A)$. Then:
$\underbrace{A}_{\text{generators}}$, $\underbrace{\mathcal{R}}_{\text{relations}}$

$$\langle A \mid \mathcal{R} \rangle := \mathrm{Free}(A) / N_{\mathcal{R}} \quad \text{is a group.}$$

We say a group G is presented by generators A & relations $\mathcal{R}$ if $G = \langle A \mid \mathcal{R} \rangle$

§7.3 Group Homomorphisms:

Definition: Let G_1, G_2 be two groups. A (set) map $f : G_1 \longrightarrow G_2$ is said to be a group homomorphism if $\forall x, y \in G$

$$f(x \underbrace{*_1}_{\text{group operation of } G_1} y) = f(x) \underbrace{*_2}_{\text{group operation of } G_2} f(y)$$

Lemma: If $f_1 : G_1 \longrightarrow G_2$ is a group homomorphism, then

(1) $f(e_1) = e_2$ (identity elements are preserved)

(2) $f(x^{-1}) = (f(x))^{-1}$ $\forall x \in G_1$ (inverse are preserved)

Proof: (1) Take $x = y = e_1$, then $f(e_1) = f(e_1 *_1 e_1) = f(e_1) *_2 f(e_1)$. So, $f(e_1) = e_2$.

(2)
$$f(x) *_2 f(x^{-1}) = f(x *_1 x^{-1}) = f(e_1) = e_2 \quad \text{by (1)}$$
$$f(x^{-1}) *_2 f(x) = f(x^{-1} *_1 x) = f(e_1) = e_2 \quad \text{by (1)}$$

So $f(x)^{-1} = f(x^{-1})$ by definition. $\square$

Definition: A group homomorphism $f : G_1 \longrightarrow G_2$ is called an isomorphism if there exists a group homomorphism $g : G_2 \longrightarrow G_1$ such that

$$f(g(x)) = x \quad \forall x \in G_2$$

and

$$g(f(y)) = y \quad \forall y \in G_1$$

Exercise: This is equivalent to saying f is a set bijection. In other words: $g = f^{-1}$ will automatically be a group homomorphism.

§7.4 Kernel & Image:

Definition: Let G_1, G_2 be two groups and $f: G_1 \longrightarrow G_2$ is a group homomorphism

- $\mathrm{Ker}(f) := \{ x \in G_1 \mid f(x) = e_2 \} \subseteq G_1$ (Kernel of f)
- $\mathrm{Im}(f) := \{ y \in G_2 \mid \exists x \in G_1 \text{ with } f(x) = y \} \subseteq G_2$ (Image of f)
  $= \{ f(x) \mid x \in G_1 \}$

Lemma: (1) $\mathrm{Ker}(f)$ is a normal subgroup of G_1

(2) $\mathrm{Im}(f)$ is a subgroup of G_2.

Proof: (1) • $e_1 \in \mathrm{Ker}(f)$ because $f(e_1) = e_2$

• $x, y \in \mathrm{Ker}(f)$, then $e_2 = f(x) = f(y)$. Then
$f(x *_1 y) = f(x) *_2 f(y) = e_2 *_2 e_2 = e_2$, so $x *_1 y \in \mathrm{Ker}(f)$

• $x \in \mathrm{Ker}(f)$, then $f(x^{-1}) = f(x)^{-1} = e_2^{-1} = e_2$, so $x^{-1} \in \mathrm{Ker}(f)$

Conclude: $\mathrm{Ker}(f) \leq G_1$

• $x \in G_1$, $y \in \mathrm{Ker}(f)$, then $xyx^{-1} \in \mathrm{Ker}(f)$ because

$$f(xyx^{-1}) = f(x) \underset{e_2}{\underset{\parallel}{f(y)}} f(x)^{-1} = f(x) f(x)^{-1} = e_2 .$$

So $x \,\mathrm{Ker}(f)\, x^{-1} \subseteq \mathrm{Ker}(f)$ for all $x \in G_1$.
By symmetry $y \,\mathrm{Ker}(f)\, y^{-1} \supseteq \mathrm{Ker}\, f$ $\forall y$ (use $x = y^{-1}$ above)
$\Rightarrow x\,\mathrm{Ker}(f)\,x^{-1} = \mathrm{Ker}(f)$. $\forall x \in G_1$

Conclude: $\mathrm{Ker}(f) \trianglelefteq G_1$

(2) • $e_2 = f(e_1) \in \mathrm{Im}(f)$

• If $y_1 = f(x_1)$, $y_2 = f(x_2)$ in $\mathrm{Im}(f)$, then $y_1 * y_2 = f(x_1 * x_2) \in \mathrm{Im}(f)$

• If $y = f(x)$, then $y^{-1} = (f(x))^{-1} = f(x^{-1})$, so $y^{-1} \in \mathrm{Im}(f)$.

Conclude: $\mathrm{Im}(f) \leq G_2$.

Q: How hard is it to check if a group homomorphism is an isomorphism?

Easy Lemma: Fix $f: G_1 \longrightarrow G_2$ group homomorphism

(1) f is one-to-one (or injective) if, and only if, $\operatorname{Ker}(f) = \{e_1\}$

(2) f is onto (or surjective) if, and only if, $\operatorname{Im}(f) = G_2$

Proof: (1) ($\Rightarrow$) Pick $x \in \operatorname{Ker}(f)$. Then $f(x) = e_2 = f(e_1)$

Since f is injective, then $x = e_1$. **Conclude:** $\operatorname{Ker}(f) \subseteq \{e_1\}$

Since $e_1 \in \operatorname{Ker}(f)$, we get $\operatorname{Ker}(f) = \{e_1\}$.

($\Leftarrow$) Fix $x, y \in G_1$ with $f(x) = f(y)$. Then:

$e_2 = f(x)\,f(y)^{-1} = f(x)\,f(y^{-1}) = f(x y^{-1})$ so $x y^{-1} \in \operatorname{Ker}(f) = \{e_1\}$.

This gives $x y^{-1} = e_1$ ie $x = y$. **Conclude:** f is injective.