Lecture VIII: More on Group homomorphisms

Recall: • A group homomorphism $f: G_1 \longrightarrow G_2$ is a set map satisfying

$$f(xy) = f(x)\, f(y) \qquad \forall\, x, y \in G_1$$

- $\mathrm{Ker}(f) = \{ x \in G_1 \mid f(x) = e_2 \}$ (Kernel of f)
- $\mathrm{Im}(f) = \{ f(x) \mid x \in G_1 \}$ (Image of f)
- Isomorphism = bijective group homomorphism

Definition: G_1, G_2 groups are isomorphic if $\exists\, f: G_1 \longrightarrow G_2$ group isomorphism.

Properties: (1) $\mathrm{Ker}(f) \trianglelefteq G_1$

(2) f is 1-to-1 (or injective) $\Longleftrightarrow \mathrm{Ker}(f) = \{e_1\}$

(3) $\mathrm{Im}(f) \leq G_2$ (in general, it's NOT normal)

§8.1 Examples:

① $\langle a \mid a^k = e \rangle \cong \mathbb{Z}/k\mathbb{Z}$ via $a^l \longmapsto \bar{l}$

② For a normal subgroup $N \trianglelefteq G$:

$$\begin{array}{ccc} G & \xrightarrow{\pi} & G/N \\ \rotatebox{90}{$\in$} & & \\ g & \longmapsto & gN =: \bar{g} \end{array}$$

is a group homomorphism. We call it the natural or canonical projection onto the quotient group

Note: $\pi(g_1 g_2) = \pi(g_1) * \pi(g_2)$ by the way in which we defined the product on G/N

③ $$\begin{array}{ccc} G_1 = GL_2(\mathbb{R}) & \xrightarrow{\det} & G_2 = \mathbb{R}_{\neq 0} \\ \rotatebox{90}{$\in$} & & \\ A & \longmapsto & \det(A) \end{array}$$

is a group homomorphism.

G_1 & G_2 are groups under standard matrix & usual multiplication, respectively.

$\mathrm{Ker}(\det) = SL_2(\mathbb{R}) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : ad - bc = 1 \right\} \trianglelefteq GL_2(\mathbb{R})$ by §6.4 Example 4.

§8.2 Building group homomorphisms:

Q: What would it take to define a group homomorphism $f: \mathrm{Free}(A) \longrightarrow H$ where H is an arbitrary group?

A: • Specify $f(a) \in H$ for every $a \in A$ (and do it as you wish, because "Free = Nothing to check")

• $w \in \text{Free}(A) \Rightarrow w = x_1^{n_1} \cdots x_\ell^{n_\ell}$ uniquely $n_1, \dots n_\ell \in \mathbb{Z}_{\neq 0}$ $x_1, \dots x_\ell \in A$
$x_1 \neq x_2,\ x_2 \neq x_3, \dots$

so $f(w) = f(x_1)^{n_1} \cdots f(x_\ell)^{n_\ell}$ is the only possible definition for $f(w) \in H$ (uniqueness of the expression of w means assignment $f(w)$ is unambiguous!)

Proposition: $\left\{ \begin{array}{c} \text{Group homomorphisms} \\ \text{Free}(A) \longrightarrow H \end{array} \right\} \xleftrightarrow{\text{1-to-1}} \left\{ \begin{array}{c} \text{Set Maps} \\ A \longrightarrow H \end{array} \right\}$

Q: Let $G = \langle A \mid \mathcal{R} \rangle$ (group presentation, as in §7.2). What would it take to define a group homomorphism $f: G \longrightarrow H$ where H is an arbitrary group?

A: • Specify $f(a) \in H$ $\forall a \in A$ $\rightsquigarrow$ $\tilde{f}: \text{Free}(A) \longrightarrow H$ group hom

• Make sure $\tilde{f}(r) = e$ $\forall r \in \mathcal{R} \subseteq \text{Free}(A)$

Proposition 2: $\left\{ f: \langle A \mid \mathcal{R} \rangle \underset{\text{gp hom}}{\longrightarrow} H \right\} \xleftrightarrow{\text{1-to-1}} \left\{ \begin{array}{c} \tilde{f}: \text{Free}(A) \longrightarrow H \text{ gp hom} \\ \mathcal{R} \subseteq \text{Ker}\,\tilde{f} \end{array} \right\}$

Proof: $\mathcal{R} \subseteq \text{Ker}\,\tilde{f}$ & $\text{Ker}\,\tilde{f} \trianglelefteq G$, so $N_{\mathcal{R}} \subseteq \text{Ker}\,\tilde{f}$. To finish the argument, we'll need the First Iso Theorem (lecture 9).

Recall $D_{2n} = \langle s_1, s_2 \mid s_1^2 = s_2^2 = (s_1 s_2)^n = e \rangle$

Example 1: Prove that there is a group homomorphism $f: D_{2n} \longrightarrow \{\pm 1\}$ with $f(s_1) = -1$, $f(s_2) = -1$.

Proof: We have to make sure $f(s_1) = -1 = f(s_2)$ preserve the relations defining D_{2n}.

$f(s_1)^2 = 1 = f(s_2)^2 = (f(s_1) f(s_2))^n$ check out because $(-1)^2 = 1 = ((-1)(-1))^n$.

$\text{Ker}(f) = \{\text{words in } s_1, s_2 \text{ of even length}\}$
$\supseteq \{e, s_1 s_2, (s_1 s_2)^2, \dots, (s_1 s_2)^{n-1}\} = \langle s_1 s_2 \rangle = H$

$H \trianglelefteq D_{2n}$ by §7.1 Ex. 2

$$\left(\begin{array}{l} D_{2n}/H = \{H, s_1 H\} \\ \quad\quad\quad\ \ \| \quad\ \ \| \leftarrow \text{since } |D_{2n}/H| = \frac{2n}{n} = 2 \\ H\backslash D_{2n} = \{H, Hs_1\} \quad \& \ s_1 \notin H. \end{array} \right)$$

Example: $H = \{e, (12)(34), (13)(24), (14)(23)\} \trianglelefteq S_4$ by §6.9 Ex 1.

S_4/H has $4!/4 = 6$ elements.

Show that: $S_4/H \simeq D_6 \quad (\simeq S_3)$

Pf/ We define $D_6 \xrightarrow{\varphi} S_4/H$ by proposing values $\varphi(r)$ & $\varphi(s)$ so that the relations in D_6 are valid for their images under φ, ie

$(\varphi(s))^2 = e = (\varphi(r))^2$ & $(\varphi(sr))^2 = e$ $\quad (s^2 = e = r^3 = (sr)^2)$

Set $s \longmapsto (1234)H$ & $r \longmapsto (123)H$

Check the relations:

$(\varphi(s))^2 = e$ because $(1234)^2 = (13)(24) \in H \quad (1234) \notin H$

$(\varphi(r))^3 = e$ because $(123)^2 = (132) \notin H$, $(123) \notin H$

$(\varphi(sr))^2 = e$ because $(1234)H\,(123)H = (1234)(123)H = (1324)H$ has order 2

$\Rightarrow \varphi(s^a r^b) = \varphi(s)^a \varphi(r)^b = (1234)^a (123)^b H$ is a group homomorphism

Since $|S_4/H| = |D_6| = 6$, we see that φ is an isomorphism if it is surjective.

But $S_4/H \overset{(*)}{=} \{ \overset{\varphi(e)}{\overset{\shortparallel}{H}}, \overset{\varphi(s)}{\overset{\shortparallel}{(1234)H}}, \overset{\varphi(r)}{\overset{\shortparallel}{(123)H}}, \underset{(123)^{-1}H}{\underset{\shortparallel}{\overset{\varphi(r^{-1})}{\overset{\shortparallel}{(132)H}}}}, \overset{\varphi(sr)}{\overset{\shortparallel}{(1324)H}}, \overset{\varphi(sr^{-1})}{\overset{\shortparallel}{(14)H}} \}$,

so φ is surjective

(*) $H \neq$ other 5 cosets because $(1234), (123), (132), (1324), (14) \notin H$.

$(1234)H \neq (123)^{-1}H$ because $(123)(1234) = (1342) \notin H$.

$(1234)H \neq (123)H$ ——— $(123)^{-1}(1234) = (132)(1234) = (1)(2)(34) \notin H$.

$(1234)H \neq (1324)H$ ——— $(1324)^{-1}(1234) = (1423)(1234) = (132) \notin H$

$(1234)H \neq (14)H$ ——— $(14)^{-1}(1234) = (14)(1234) = (123) \notin H$

$(123)H \neq (132)H$ ——— $(123)^{-1}(132) = (132)(132) = (123) \notin H$

$(123)H \neq (1324)H$ ——— $(123)^{-1}(1324) = (132)(1324) = (1243) \notin H$

$(123)H \neq (14)H$ ——— $(123)^{-1}(14) = (132)(14) = (1432) \notin H$

$(132)H \neq (1324)H$ ——— $(132)^{-1}(1324) = (123)(1324) = (1)(24)(3) \notin H$

$(132)H \neq (14)H$ ——— $(132)^{-1}(14) = (123)(14) = (1423) \notin H$

$(1324)H \neq (14)H$ ——— $(14)^{-1}(1324) = (14)(1324) = (132) \notin H$.