Lecture IX: Three Isomorphism Theorems

§9.1 First Isomorphism Theorem:

Let $f: G_1 \longrightarrow G_2$ be a group homomorphism. Write $K := \mathrm{Ker}(f) \trianglelefteq G_1$ and let $\pi: G_1 \longrightarrow G_1/K$ be the natural projection.

Theorem: (1) There exists a unique $\bar{f}: G_1/K \longrightarrow G_2$ such that

$$f(x) = \bar{f}(\pi(x)) \quad (= \bar{f}(xK)) \quad \forall x \in G_1$$

More precisely, we have the commutative diagram:

$$\begin{tikzcd}
G_1 \arrow[r, "f"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\circlearrowleft"] & G_2 \\
G_1/K \arrow[ur, "\bar{f}"'] &
\end{tikzcd}
\qquad \bar{f}(xK) = f(x)$$

(2) $\bar{f}$ sets up an isomorphism $G_1/K \simeq \mathrm{Im}(f)$.

We get the commutative diagram

$$\begin{tikzcd}
G_1 \arrow[r, "f"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\circlearrowleft"] & G_2 \arrow[d, phantom, "\rotatebox{90}{$\leq$}"] \\
G_1/K \arrow[r, "\bar{f}", "\simeq"'] & \mathrm{Im}(f)
\end{tikzcd}$$

Example: $G_1 = GL_2(\mathbb{R}) \xrightarrow{\det} \mathbb{R}_{\neq 0}$ is a surjective group homomorphism.

$\mathrm{Ker}(\det) = \{X \in GL_2(\mathbb{R}) : \det(X) = 1\} = SL_2(\mathbb{R})$

Hence, by First Iso Thm: $GL_2(\mathbb{R})/SL_2(\mathbb{R}) \simeq \mathbb{R}_{\neq 0}$.

Proof: (1) We do not have much of a choice to define $\bar{f}: G_1/K \longrightarrow G_2$

$$\bar{f}(xK) = f(x) \qquad (\text{so} \quad \bar{f} \circ \pi = f)$$

- We need to check $\bar{f}$ is well-defined: $xK = yK \overset{?}{\Longrightarrow} f(x) = f(y)$.

  But $xK = yK \Longleftrightarrow x^{-1}y \in K = \mathrm{Ker}(f)$.

  Since $e_2 = f(x^{-1}y) = f(x)^{-1} f(y)$, we get $f(x) = f(y)$ as we wanted.

• Check 1 (easy): $\bar{f}$ is a group homomorphism.

(2) • Check 2: $\mathrm{Ker}(\bar{f}) = \{e_1 K\}$. Hence by Easy Lemma §7.4, $\bar{f}$ is injective.

Pf/ $\bar{f}(xK) = e_2 \implies f(x) = e_2 \implies x \in K$ and $xK = e_1 K$.

By definition $\mathrm{Im}(\bar{f}) = \mathrm{Im}(f) =: H_2 \leq G_2$ and

$\bar{f}: G_1/K \longrightarrow H_2 \leq G_2$ is both injective and surjective.

Since $\bar{f}$ is a group homomorphism and a bijection, we conclude $\bar{f}$ is an iso. □

§9.2 Applications:

Theorem (Classification of cyclic groups)

Any cyclic group G is isomorphic to $\mathbb{Z}$ or $\mathbb{Z}/k\mathbb{Z}$ for $k = 1, 2, \ldots$ (and only one of them)

Proof: Fix $G = \langle a \rangle$ cyclic. Define $f: \mathbb{Z} \longrightarrow G$, $1 \longmapsto a$. Since $\mathbb{Z} = \mathrm{Free}(1)$,

f is a group homomorphism.

• f is surjective by construction

• $\mathrm{Ker}(f) \leq \mathbb{Z} = (1)$, so it is also a cyclic group. $\mathrm{Ker}(f) = \langle k \rangle$ for some $k \geq 0$.

Then, by First Isomorphism Theorem, we have $\mathbb{Z}/\mathrm{ker}\, f \simeq G$.

Here, $\mathbb{Z} = \mathbb{Z}/(0) = \mathbb{Z}/0\mathbb{Z}$. if $\mathrm{Ker}\, f = (0)$

The answers are all non-isomorphic by cardinality. □

• Another way to interpret (or use) First Iso Theorem:

Lemma: Let $f: G_1 \longrightarrow G_2$ be a surjective group homomorphism and let $H \trianglelefteq G$ be such that $f(x) = e_2$ $\forall x \in H$ (ie $H \subseteq \mathrm{Ker}(f)$). Then:

$$H = \mathrm{Ker}(f) \iff G_1/H \cong G_2 \text{ with } \cong \text{ induced from } f.$$

Proof: ($\Rightarrow$) is First Iso Theorem

($\Leftarrow$) Let $G_1 \xrightarrow{\pi} G_1/H$ be the natural projection. Since

$$\begin{tikzcd} G \arrow[r, "\pi"] \arrow[dr, "f"'] & G_1/H \arrow[d, "\wr\varphi"] \\ & G_2 \end{tikzcd}$$

(i.e., $\varphi \circ \pi = f$ & φ is an iso) we get $H = \ker(\pi) = \operatorname{Ker}(f)$ $\square$

Back to Example 1 §8.2: $D_{2n} \xrightarrow{f} \{\pm 1\}$ $f(s_1) = f(s_2) = -1$, $\ker(f) \supseteq H = \langle s_1 s_2 \rangle$

$D_{2n} = \langle s_1, s_2 \mid s_1^2 = s_2^2 = (s_1 s_2)^n = e \rangle$ $H \trianglelefteq D_{2n}$

Now: $D_{2n}/H \xrightarrow{\cong} \{\pm 1\}$ because the map is surjective & $|D_{2n}/H| = 2$

$\bar{w} \longmapsto f(w)$

So by Lemma §9.2: $H = \operatorname{Ker}(f)$

Proposition 2 §8.2: $\{f : \langle A \mid \mathcal{R} \rangle \longrightarrow H \text{ gp hom}\} \xleftrightarrow{1\text{-to-}1} \{\tilde{f} : \operatorname{Free}(A) \longrightarrow H \text{ gp hom}, \ \mathcal{R} \subseteq \operatorname{Ker} \tilde{f}\}$

Proof: • Fix $\tilde{f}$ on (RHS). Now, $\mathcal{R} \subseteq \operatorname{Ker} \tilde{f} \trianglelefteq \operatorname{Free}(A)$ so $N_{\mathcal{R}} \subseteq \ker \tilde{f}$

Define $f(w N_{\mathcal{R}}) = \tilde{f}(w)$. It is well-defined because $N_{\mathcal{R}} \subseteq \operatorname{Ker} \tilde{f}$.

• Conversely $f : \operatorname{Free}(A)/N_{\mathcal{R}}$ defines $\tilde{f} : \operatorname{Free}(A) \longrightarrow H$ by $\tilde{f}(a) = f(\bar{a})$.

$\tilde{f}(r) = 0$ for all $r \in \mathcal{R}$ so $\mathcal{R} \subseteq \operatorname{Ker} \tilde{f}$.

• By construction the correspondence is 1-to-1. (f is uniquely determined by $\tilde{f}$. & conversely). $\square$

Example: Conjugation by $a \in G$.

$\operatorname{Conj}(a)$ or $\operatorname{Ad}(a)$: $G \longrightarrow G$ is a group homomorphism (check)

$g \longmapsto aga^{-1}$

notation from Representation Theory

• $\operatorname{Ker}(\operatorname{Conj}(a)) = \{e\}$ $aga^{-1} = e \iff g = a^{-1}ea = e$

• $\operatorname{Im}(\operatorname{Conj}(a)) = G$ because $g = aa^{-1}gaa^{-1} = \operatorname{Conj}(a)(a^{-1}ga)$ $\forall g \in G$.

So $\operatorname{Conj}(a)$ is an iso $\forall a \in G$.

In fact $\mathrm{Conj}(ab) = \mathrm{Conj}(a) \circ \mathrm{Conj}(b)$ composition

$\mathrm{Conj}(e) = \text{identity} \quad \begin{array}{ccc} G & \longrightarrow & G \\ x & \longmapsto & x \end{array}$

Hence: $(\mathrm{conj}(a))^{-1} = \mathrm{conj}(a^{-1})$

Example: $G_1 = \mathrm{Free}(2) = \langle a, b \mid \text{no rels} \rangle \xrightarrow{\quad p \quad} G_2 = \mathbb{Z}^2$

$\quad w \in G_1 \longmapsto (\text{end point of } \gamma_w) = (\#a\text{'s}, \#b\text{'s})$

E.g.: $p(a^2 b^2 a^{-7}) = p(a^{-5} b^2) = (-5, 2)$.

- Claim 1: p is surjective $\quad (n_1, n_2) = p(a^{n_1} b^{n_2})$

- Set $H := \langle xyx^{-1}y^{-1} \mid x, y \in \mathrm{Free}(2) \rangle$ (from §9.3)

- Claim 2: $H \subseteq \mathrm{Ker}(p)$

Pf/ $\gamma(xyx^{-1}y^{-1}) = 0$ since γ_w ends at $(0,0)$ for all $w = xyx^{-1}y^{-1}$.

$\Rightarrow \gamma_{(H)} = \langle \gamma(xyx^{-1}y^{-1}) \mid x, y \in \mathrm{Free}(2) \rangle \subseteq \mathrm{Ker}(p)$.

- Claim 3: $H \trianglelefteq G_1$.

Assuming the claim is true, we get $G_1/H = \langle \bar{a}, \bar{b} \mid \bar{a}\bar{b} = \bar{b}\bar{a} \rangle$ because $aba^{-1}b^{-1} = ab\,(ba)^{-1} \in H$.

Since $\begin{array}{ccc} G_1/H & \overset{\bar{p}}{\cong} & \mathbb{Z}^2 \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ \bar{a}^m \bar{b}^n & \longmapsto & (m,n) \end{array}$, the Lemma implies $H = \mathrm{Ker}(p)$. $\square$

Q: Why is Claim 3 true?

A: $H = [G:G] \trianglelefteq G$ is the commutator of G (see HW2)

§9.3 Second and Third Isomorphism Theorems:

Note: The textbook calls them 3rd and 4th Iso Theorems.

Fix G a group and $N \trianglelefteq G$ a normal subgroup. Consider $\pi : G \longrightarrow G/N$ the natural projection

Theorem 2: π sets up an order/ index preserving bijection

$$\left\{\begin{matrix} \text{Subgroups } H \leq G \\ \text{s.t } N \subseteq H \end{matrix}\right\} \longleftrightarrow \left\{ \text{Subgroups } \bar{H} \leq G/N \right\}$$

$$\cup\!\!\!\mid \qquad\qquad\qquad\qquad \cup\!\!\!\mid$$

$$\left\{\begin{matrix} \text{normal subgroups } A \trianglelefteq G \\ \text{st. } N \subseteq A \end{matrix}\right\} \longleftrightarrow \left\{ \text{normal subgroups } \bar{A} \leq G/N \right\}$$

Correspondences:

$$A \longmapsto \pi(A)$$

$$\pi^{-1}(\bar{A}) \longleftarrow\!\shortmid \bar{A}$$

Proof: Once the correspondences are specified, the proof that they are well-defined and they are mutual inverse maps is a routine check. It requires 2 lemmas, whose proofs are left as exercises.

Lemma 1: $f: G_1 \longrightarrow G_2$ group homomorphism. Then

(1) $H_1 \leq G_1 \implies f(H_1) \leq G_2$

(2) $H_2 \leq G_2 \implies f^{-1}(H_2) = \{ x \in G_1 : f(x) \in H_2 \} \leq G_1$.

(3) $N_2 \trianglelefteq G_2 \implies f^{-1}(N_2) \trianglelefteq G_1$

Lemma 2: If $f: G_1 \longrightarrow G_2$ is a surjective group homomorphism, and $N_1 \trianglelefteq G_1$, then $f(N_1) \trianglelefteq G_2$.

Theorem 3: Given $N \trianglelefteq G$, let $H \trianglelefteq G$ with $N \subseteq H$. Then:

(1) $H/N \trianglelefteq G/N$

(2) $G/H \simeq \dfrac{G/N}{H/N}$

Proof: Consider $\pi: G \longrightarrow G/N$. Then $H/N = \pi(H) \trianglelefteq G/N$ by Lemma 2 because π is surjective. This shows (1)

For (2), we consider the projection $\pi_2: G/N \longrightarrow \dfrac{G/N}{H/N}$. Then, we have

$\varphi = \pi_2 \circ \pi : G \longrightarrow \dfrac{G/N}{H/N}$ group homomorphism.

- The map φ is surjective because both π & π_2 are surjective.
- $\ker\varphi = \{x \in G : \pi_2(\pi(x)) = e^{H/N}\} = \{x \in G : \pi(x) \in \ker \pi_2 = H/N\}$
$$= \pi^{-1}(H/N) \underset{(*)}{=} H$$

(*) ($\supseteq$) is clear

($\subseteq$) $\pi(x) = xN \in H/N \iff xN = hN$ for some $h \in H$. But this means $h^{-1}x \in N \subseteq H$, so $x \in hH \subseteq H$. We conclude that $x \in H$. □

$\Rightarrow$ By 1st Iso Theorem $\overline{\varphi} : G/H \xrightarrow{\sim} (G/N)/(H/N)$ □