Lecture X: Group actions on a set

Summary of the Course (Weeks 1 & 2) So far, we have

(1) Defined useful terms from Group Theory:

- Groups, subgroups, subgroups generated by a set, order of a group
- Normal subgroups, normal subgp ________
- Left and Right cosets (G/H & $H\backslash G$), Quotient groups
- Group Homomorphisms / Isomorphisms; kernel & Image of gp homomorphisms.
- Group presentations

(2) Examples: Free groups, D_{2n}, S_n

(3) Main results: 3 Isomorphism Thms, Classification of cyclic groups.

TODAY: Group actions on Sets.

§10.1 Definition:

Fix G group and X any set.

Definition: An action of G on X is a set map $G \times X \xrightarrow{\alpha} X$ satisfying

$$(g, x) \longmapsto \alpha(g,x) =: g \cdot x$$

(1) $e \cdot x = x \quad \forall x \in X$

(2) $(g_1 g_2) \cdot x = g_1 \cdot (g_2 \cdot x) \quad \forall g_1, g_2 \in G$ and $x \in X$.

Given α as above, we say G acts on X, we sometimes denote this by $G \circlearrowright_\alpha X$ if we want to specify α

Remark: Giving such α is the same as specifying a group homomorphism

$$\Psi: G \longrightarrow \mathrm{Aut}_{\mathrm{Set}}(X) := \{f: X \to X \mid f \text{ is a bijection}\}$$
$$g \longmapsto \alpha_g : x \longmapsto g \cdot x$$

$$\alpha(g,x) = \Psi(g)(x)$$

Note $\mathrm{Aut}_{\mathrm{Set}}(X)$ is a group under composition (set-automorphisms of X, or symmetric group on X)

(1) means $\alpha_e = \mathrm{id}_X \rightsquigarrow$ neutral in $\mathrm{Aut}(X)$

(2) $\alpha_{g_1 g_2} = \alpha_{g_1} \circ \alpha_{g_2}$ means Ψ is multiplicative.

- α_g is a bijection because $\alpha_{g^{-1}} \circ \alpha_g = \alpha_{g^{-1}g} = \underset{\substack{\shortparallel \\ id_X}}{\alpha_e} = \alpha_g \circ \alpha_{g^{-1}}$, meaning $(\alpha_g)^{-1} = \alpha_{g^{-1}}$.

A word on notation: • Aut is short for automorphisms (meaning, isomorphisms with self)
• Subscript: "structure" that we want to preserve

⚠ These actions are often referred to as "left-actions" since there is a notion of right-action as well. For a right-action we need $\begin{array}{ccc} G \times X & \xrightarrow{\alpha} & X \\ (g,x) & \longmapsto & x \cdot g \end{array}$ to satisfy

(1) $x \cdot e = e \quad \forall e$

(2') $x \cdot g_1 g_2 = (x \cdot g_1) \cdot g_2$

We will write $X \circlearrowleft G$ when we are using right-actions.

Examples: $G \circlearrowright G/H$ vs $H \backslash G \circlearrowleft G$

$g \cdot xH = gxH$ $\qquad$ $Hx \cdot g = Hxg$

- Check this is well-defined (independent of the coset representative)
- Check the 2 axioms defining left and right action.

Example: $G = GL_n(\mathbb{R}) \circlearrowright X = \mathbb{R}^n$ by $\begin{array}{ccc} G \times X & \xrightarrow{\alpha} & X \\ (A, \vec{v}) & \longmapsto & A\vec{v} \end{array}$

So α_A = matrix multiplication by A.

Check: (1) $I_d \cdot \vec{v} = \vec{v} \quad \forall \vec{v} \in \mathbb{R}^n$

(2) $(AB) \cdot \vec{v} = AB\vec{v} = A(B\vec{v}) = A \cdot (B\vec{v}) = A \cdot (B \cdot \vec{v}) \quad \forall A, B \in G$
$\forall \vec{v} \in X$.

In this case $G \longrightarrow \mathrm{Aut}_{\text{Set}}\, X$ has image in $\mathrm{Aut}_{\text{v.sp}}(\mathbb{R}^n)$ (because α_A preserves the vector space structure of $\mathbb{R}^n$.

§10.2 More definitions

Fix $G \circlearrowright X$

Definition: The orbit of an element $x \in X$ is the following subset of X:

$$G \cdot x = \{ g \cdot x \mid g \in G \}$$

Definition: The stabilizer of $x \in X$ is the following subgroup of G:

$$\mathrm{Stab}_G\, x = \{ g \in G \mid g \cdot x = x \} \leq G$$

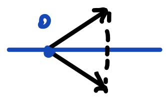
⚠
- Typically, $\mathrm{Stab}_G\, x \ntrianglelefteq G$
- By construction, it is a subgroup.

Definition: The fixed point set of $g \in G$ is the following subset of X:

$$X^g := \{ x \in X \mid g \cdot x = x \} \subseteq X$$

Example: Consider the group homomorphism

$$D_{2n} \xrightarrow{f} GL_2(\mathbb{R}) \subseteq \mathrm{Aut}_{set}(\mathbb{R}^2)$$
$$s \longmapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$
$$r \longmapsto \begin{bmatrix} \cos\frac{2\pi}{n} & -\sin\left(\frac{2\pi}{n}\right) \\ \sin\frac{2\pi}{n} & \cos\left(\frac{2\pi}{n}\right) \end{bmatrix}$$

Note: $\Psi(s) =$  reflection about x-axis

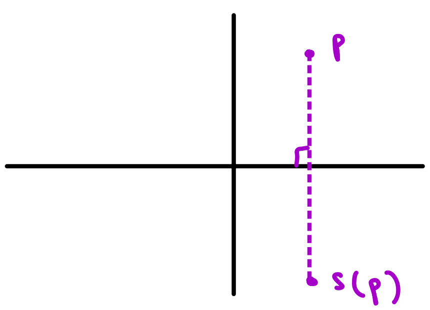
$\Psi(r) =$ rotation of $\measuredangle = \frac{2\pi}{n}$

Since $f(s)^2 = f(r)^n = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ & $f(s)f(r)f(s) = f(r^{-1})$ by construction, we get that f is a group homomorphism. Thus, it defines $D_{2n} \circlearrowright \mathbb{R}^2$

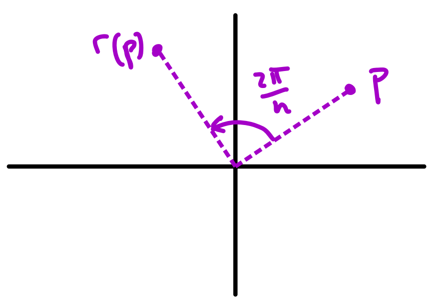
Note: $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is fixed by all elements of D_{2n}, so we can omit it and consider

$$D_{2n} \circlearrowright X = \mathbb{R}^2 \setminus \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$$

- Let's compute the orbit of a point $p \in X$:

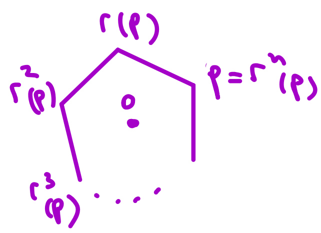


s = reflection about x-axis



r = rotation by $2\pi/n$

$\rightsquigarrow$



$\|$

orbit of a point $p \in X = \mathbb{R}^2 \setminus \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$ under $\langle r \rangle$ (= vertices of a regular n-gon)

$$\Rightarrow D_{2n} \cdot p \supseteq \underbrace{\{ p, r(p), r^2(p), \ldots, r^{n-1}(p) \}}_{\text{all distinct}}$$

s (n-gon) will reflect the n-gon about the x-axis.

Claim: $|D_{2n} \cdot p| = 2n \iff s(p) \notin \{p, r(p), \dots, r^{n-1}(p)\}$

Proof: ($\Rightarrow$) If $s(p) = r^k(p)$ for some $k = 0, \dots, n-1$ then we have $|D_{2n} \cdot p| < 2n$, which is a contradiction.

($\Leftarrow$) We prove the contrapositive statement.

If $|D_{2n} \cdot p| < 2n$, then we have a repeated element x.

Claim: $x = sr^k(p) = r^j(p)$ for some k, j

Pf/ $\{p, r(p), \dots, r^{n-1}(p)\}$ are all distinct

$$\Downarrow$$

$\{s(p), sr(p), \dots, sr^{n-1}(p)\}$ ——— $(sr^k(p) = sr^j(p) \Rightarrow r^k(p) = r^j(p))$

So the repetition can only come from some x in both sets $\square$

- But using $sr^k = r^{-k}s$ in D_{2n}, we get $r^j(p) = sr^k(p) = r^{-k}s(p)$,

so $r^{k+j}(p) = r^k r^j(p) = s(p) \Rightarrow s(p) \in \{p, r(p), \dots, r^{n-1}(p)\}$