Lecture XI: Orbit-Stabilizer Correspondence

Recall: G a group, X a set.

A (left) G-action on X is a set map $G \times X \longrightarrow X$, $(g, x) \longmapsto g\cdot x$ st

(1) $e \cdot x = x \quad \forall x$

(2) $g_1 \cdot (g_2 \cdot x) = (g_1 g_2)\cdot x \quad \forall g_1, g_2 \in G \quad \forall x$

(Equivalently: we need a group homomorphism $G \longrightarrow \mathrm{Aut}_{Set}(X)$)

- Given $x \in X$ ⇝ Orbit of x $= G\cdot x \subseteq X$
  ⇝ Stabilizer of x $\mathrm{Stab}_G(x) = \{ g \in G : g\cdot x = x\} \leq G$
- $X^g := \{ x : g \cdot x = x \}$ (Fix points under the action of $g \in G$)

§ 11.1 Example:

Consider the group homomorphism

$$D_{2n} \xrightarrow{f} GL_2(\mathbb{R}) \subseteq \mathrm{Aut}_{Set}(\mathbb{R}^2)$$

$$s \longmapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{reflection about } x\text{-axis}$$

$$r \longmapsto \begin{bmatrix} \cos\frac{2\pi}{n} & -\sin\left(\frac{2\pi}{n}\right) \\ \sin\frac{2\pi}{n} & \cos\left(\frac{2\pi}{n}\right) \end{bmatrix} \quad \text{rotation of } \measuredangle = \frac{2\pi}{n}$$

and the action induced on $X = \mathbb{R}^2 \setminus \{\begin{bmatrix} 0 \\ 0 \end{bmatrix}\}$

$\Rightarrow D_{2n}\cdot p \supseteq \{ p, r(p), r^2(p), \ldots, r^{n-1}(p)\}$ all distinct

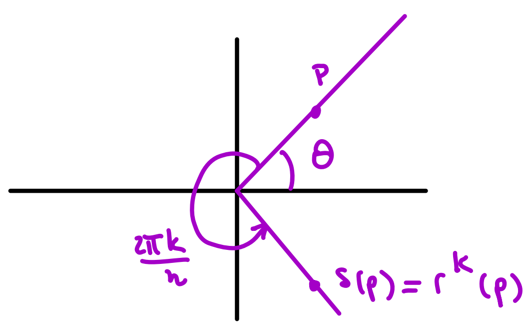
$\Rightarrow D_{2n}\cdot p \supseteq \{ s(p), sr(p), sr^2(p), \ldots, sr^{n-1}(p)\}$ all distinct

Claim: $|D_{2n}\cdot p| = 2n \iff s(p) \notin \{p, r(p), \ldots, r^{n-1}(p)\}$

Q: Which points p have $|D_{2n}\cdot p| \neq 2n$?

A: Need $s(p) = r^k(p)$ for some k, ie reflecting p about x-axis is the same as rotating by $\frac{2\pi k}{n}$

Equivalently $r^{-k}s(p) = p \iff sr^k(p) = p$.



Write $p = Re^{\theta i} = R\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$ for $0 < \theta < 2\pi$.

$$p = sr^k(p) \iff Re^{\theta i} = s\left(Re^{(\theta + \frac{2\pi k}{n})i}\right) = Re^{-(\theta+\frac{2\pi k}{n})i}$$
$$\iff \theta \equiv -(\theta + \tfrac{2\pi k}{n}) \mod (2\pi)$$
$$\iff 2\theta \equiv -\tfrac{2\pi k}{n} \mod (2\pi)$$
$$\iff \theta = -\tfrac{\pi k}{n} \mod (\pi)$$

Conclude: $|D_{2n}\cdot p| < 2n \iff p = Re^{\theta i}$ with $\theta = \frac{n-k}{n}\pi$ or $\frac{2n-k}{n}\pi$ for $0 \leq k \leq n$

In short: $p = R\begin{bmatrix}\cos(\frac{n-k}{n}\pi)\\ \sin(\frac{n-k}{n}\pi)\end{bmatrix} = R\begin{bmatrix}-\cos(\frac{k\pi}{n})\\ \sin(\frac{k\pi}{n})\end{bmatrix}$ or $p = R\begin{bmatrix}\cos\frac{k\pi}{n}\\ -\sin\frac{k\pi}{n}\end{bmatrix}$

In these situations: $|D_{2n}\cdot p| = n$ since $D_{2n}\cdot p = \{p, r(p), \dots, r^{n-1}(p)\}$

Q: $\mathrm{Stab}_{D_{2n}}(p) = ?$

A: If $|D_{2n}\cdot p| = 2n$, then $\mathrm{Stab}_{2n}(p) = \{id\}$

Otherwise, $\langle sr^k\rangle \subseteq \mathrm{Stab}_{D_{2n}}(p)$ if $s(p) = r^k(p)$

Claim: Equality holds

Pf/ $r^j \notin \mathrm{Stab}_{D_{2n}}(p)$ for all $j = 1, \dots, n-1$. $\Rightarrow \mathrm{Stab}_{D_{2n}}(p) \subseteq \{s, sr, \dots, sr^{n-1}\}$

If $sr^j(p) = p = sr^k(p) \Rightarrow r^j(p) = r^k(p) \Rightarrow r^{j-k}(p) = p$, forcing $\bar{j} \equiv \bar{k} \mod n$ ie $j = k$.

Thus $\mathrm{Stab}_{D_{2n}}(p) \subseteq \langle sr^k\rangle$.

- Alternative argument: equality holds since $|\mathrm{Stab}_{D_{2n}}(p)| = \dfrac{|D_{2n}|}{|D_{2n}\cdot p|} = \dfrac{2n}{n} = 2$.
  and $(sr^k)^2 = 1$.

Note $\langle sr^k\rangle \ntrianglelefteq D_{2n}$ in general, so orbit stabilizers need not be normal!

Theorem (Orbit-Stabilizer Correspondence) Let $G \curvearrowright X$

(1) For every $x \in X$, we have a (set) bijection

$$\begin{aligned} G/\mathrm{Stab}_G(x) &\xrightarrow{\Phi} G\cdot x \\ \bar{g} &\longmapsto g\cdot x \end{aligned}$$

(2) For every $\sigma \in G$ and $x \in X$ we have an isomorphism of groups :

$$\begin{array}{ccc} \mathrm{Stab}_G(x) & \longrightarrow & \mathrm{Stab}_G(\sigma\cdot x) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ g & \longmapsto & \sigma g \sigma^{-1} \end{array}$$

Proof: (1) If $h \in \mathrm{Stab}_G(x)$, $gh\cdot x = g\cdot(h\cdot x) = g\cdot x$, so Φ is well-defined. We check it's both injective and surjective.

- By definition, this map is surjective.
- If $g_1\cdot x = g_2\cdot x$, then $\underset{\displaystyle (g_2^{-1}g_1)\cdot x}{g_2^{-1}\cdot(g_1\cdot x)} = g_2^{-1}\cdot(g_2\cdot x) = (g_2^{-1}g_2)\cdot x = e\cdot x = x$

yields $g_2^{-1}g_1 \in \mathrm{Stab}_G(x)$, ie $\overline{g_1} = \overline{g_2}$ in $G/\mathrm{Stab}_G X$.

(2) $g \in \mathrm{Stab}_G(x) \iff g\cdot x = x \iff \sigma\cdot((g\sigma^{-1}\sigma)\cdot x) = \sigma\cdot x$

$\sigma\cdot((g\sigma^{-1}\sigma)\cdot x) = (\sigma g\sigma^{-1})\cdot(\sigma\cdot x)$

$\iff \sigma g \sigma^{-1} \in \mathrm{Stab}_G(\sigma\cdot x)$

Since $\mathrm{Conj}(\sigma)$ is an isomorphism of groups, we have proved the desired statement. □

Example: $S_n \curvearrowright \{1,2,\dots,n\} =: X$ $\quad |X^\sigma| = \#$ of 1-cycles in σ

$\sigma \in S_n$

(Eg: $n=5$ $\quad X^{(123)(4)(5)} = \{4,5\}$)

By construction $X =$ disjoint union of orbits under σ.

$\longleftrightarrow$ writing σ as a product of disjoint cycles

- $\mathrm{Stab}_G(k) = \{\pi \in S_n \mid \pi(k) = k\} \simeq \{\pi' \in S_n \mid \pi(n) = n\} = \mathrm{Stab}_G(n)$

$(1 \leq k \leq n)$ $\qquad \omega \longmapsto (kn)\,\omega\,(kn)^{-1} = (kn)\,\omega\,(kn)$

$n = (kn)\cdot k$

So $\mathrm{Stab}_G(k) \simeq S_{n-1} \leq S_n$

§ 11.3 A fun Application :

Recall: $S_n \circlearrowright \{1, \dots, n\}$ transitive action (one orbit)

We can extend the action to partitions of $\{1, \dots, n\}$ of fixed length

$$X := \{ P_1 \sqcup \cdots \sqcup P_r = \{1, \dots, n\} \quad \text{length}(P_i) = \lambda_i \quad , \lambda_1 + \dots + \lambda_r = n \}$$
$$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r$$

by $\sigma \cdot (P_1 \sqcup \cdots \sqcup P_r) = (\sigma \cdot P_1) \sqcup (\sigma \cdot P_2) \sqcup \cdots \sqcup (\sigma \cdot P_r)$

Note: Size of P_i & $\sigma \cdot P_i$ matches. $\text{length}(P_i) = \lambda_i = \text{length}(\sigma \cdot P_i)$

By construction: the action is transitive on partitions of the same length

Eg: $x_0 = \{1,2,3\} \sqcup \{4,5\} \sqcup \{6\}$, $(\lambda_1 = 3, \lambda_2 = 2, \lambda_1 = 1)$ partition of 6

$\text{Stab}_{S_7}(\{1,2,3\} \sqcup \{4,5\} \sqcup \{6\}) \cong S_3 \times S_2 \times S_1$ so $|X| = \dfrac{|S_6|}{|\text{Stab}_G(x_0)|} = \dfrac{6!}{3!\,2!\,1!}$
by Theorem (1).