Lecture XII: Applications & Counting lemmas

Theorem: (1) $G/\mathrm{Stab}_G(x) \longrightarrow G\cdot x$ is a bijection

$$\bar{g} \longmapsto g\cdot x$$

(2) $\mathrm{Stab}_G(x) \xrightarrow{\sim} \mathrm{Stab}_G(\sigma\cdot x)$ given $\sigma \in G$, $x \in G$

$$g \longmapsto \sigma\cdot g\sigma^{-1}$$

Theorem: Fix a group G, a set X and an action $G \circlearrowright X$. For any $\sigma, g \in G$ we have a bijection $X^{g} \xrightarrow{\sim} X^{\sigma g \sigma^{-1}}$

$$x \longmapsto \sigma\cdot x$$

Proof: The map is a restriction of the bijection $X \xrightarrow{\sigma\cdot -} X$. Thus, we only need to check that the map restricts to the fixed point sets.

$$x \in X^{g} \iff g\cdot x = x \iff g\sigma^{-1}\cdot(\sigma\cdot x) = x$$
$$\iff \sigma\cdot(g\sigma^{-1})\cdot(\sigma\cdot x) = \sigma\cdot x$$
$$(\sigma g\sigma^{-1}) \overset{''}{\cdot} (\sigma\cdot x) = \sigma\cdot x \iff \sigma\cdot x \in X^{\sigma g \sigma^{-1}}$$

§12.1 Fun Applications:

① Multinomial Coefficients:

Recall: $S_n \circlearrowright \{1,\dots,n\}$ has one orbit.

Consider the set of ordered partitions of $\{1,\dots,n\}$ of type $\lambda_1 \geq \dots \geq \lambda_r$

$$X = \{(P_1, P_2, \dots, P_r) \mid \bigsqcup_{i=1}^{n} P_i = \{1,\dots,n\}, \ \mathrm{length}(P_i) = \lambda_i\}$$

Note: $n=7$ $(123, 45, 67) \neq (123, 67, 45)$ as elements of X but they induce the same partition.

$S_n \circlearrowright X$ via $\sigma\cdot(P_1,\dots,P_r) = (\{\sigma(x) : x \in P_1\}, \dots, \{\sigma(x) : x\in P_r\})$

- **Claim:** The action has one orbit. $(S_n \cdot \underbrace{((1\cdots\lambda_1),(\lambda_1+1,\dots\lambda_1+\lambda_2),\dots,(\lambda_1+\dots+\lambda_{r-1}+1,\dots n))}_{=x_0} = X)$

Pf/ $P_1,\dots,P_r = \sigma\cdot x_0$ with $\sigma = [x^1_1,\dots x^1_{\lambda_1}, x^2_1,\dots x^2_{\lambda_2},\dots,x^r_1,\dots x^r_{\lambda_r}]$ in 1-line notation. $\square$

- Stabilizers?

Eg: $\mathrm{Stab}_{S_7}(123, 45, 67) = \{\sigma \in S_7 \mid \{\sigma(1),\sigma(2),\sigma(3)\} = \{1\ 2\ 3\}$, $\{\sigma(4),\sigma(5)\} = \{4,5\}$, $\{\sigma(6),\sigma(7)\} = \{6,7\}\}$

$$= S_3 \times S_2 \times S_2$$

$\mathrm{Stab}_{S_n}((P_1, P_2, \dots, P_r)) \cong S_{\lambda_1} \times S_{\lambda_2} \times \dots \times S_{\lambda_r}$

$\Rightarrow |\mathrm{Stab}_{S_n}(x_0)| = \lambda_1! \, \lambda_2! \dots \lambda_r!$

- Theorem (1) $\Rightarrow$ $|X| = |S_n \cdot x_0| = \dfrac{|S_n|}{|\mathrm{Stab}_{S_n}(x_0)|} = \dfrac{n!}{\lambda_1! \dots \lambda_r!}$

Proposition 1: # ordered partitions of type $\lambda_1 \geq \dots \geq \lambda_r$ $= \dfrac{n!}{\lambda_1! \dots \lambda_r!}$

2 **Proposition 2:** Fix $m \in \mathbb{Z}_{\geq 1}$ and $p \in \mathbb{Z}_{\geq 2}$ prime. Then,

$$\binom{p^r m}{p^r} \equiv m \mod p$$

Proof: Consider $G = \mathbb{Z}/p^r\mathbb{Z}$ group under $+$ & $X = \{x_1, \dots, x_m\}$ some set with m elements

$G \circlearrowright G$ by $\bar{g} \cdot \bar{l} = \overline{g+l}$

E = set of all p^r-element subsets of $G \times X$. $\Rightarrow |E| = \binom{p^r m}{p^r}$

Set $G \circlearrowright G \times X$ by $\sigma \cdot (g, x) = (\sigma \cdot g, x)$ (act on G-leg)

Claim 1: This induces an action on E $\sigma \cdot (\{e_1, \dots e_{p^r}\}) = \{\sigma \cdot e_1, \dots, \sigma \cdot e_{p^r}\}$

no repetitions $\Rightarrow$ no repetitions

Now, E = disjoint union of orbits. For $\mathcal{O} \in G \backslash E$, $|\mathcal{O}|$ divides $|G| = p^r$

$\Rightarrow$ either $|\mathcal{O}| = 1$ or $p \mid |\mathcal{O}|$.

Thus, $|E| \equiv$ # orbits with exactly one element (mod p)

Claim 2: # orbits with exactly one element $= m$.

Pf/ $y_0 \in E$ with $G \cdot y_0 = y_0$, Write $y_0 = \{e_1, \dots, e_{p^r}\}$ $e_i = (g_i, x_i)$ $\forall i$

But $\{\sigma \cdot e_1, \dots, \sigma \cdot e_{p^r}\} = \{(\sigma + g_1, x_{i_1}), \dots, (\sigma + g_{p^r}, x_{i_{p^r}})\} = \{(g_1, x_{i_1}), \dots, (g_{p^r}, x_{i_{p^r}})\}$ $\forall \sigma$

$\Leftrightarrow$ $x_{i_1} = \dots = x_{i_{p^r}}$ is the same element of X. (Reason: $G \cdot g_1 = G$ has size p^r and $G \times \{x_0\} \subseteq y_0$.)

So $y_0 = G \times \{x_0\}$ by size considerations

$\Rightarrow$ The number of such orbits is $|X| = m$. $\square$

§12.2 Counting lemmas:

Q: Given $G \curvearrowright X$, how many orbits are there?

Burnside's Counting Lemma: " # of orbits = average # of fixed points "

Notation: ${}_G\backslash^X$ = set of orbits (of the left action)

$= \{ \mathcal{O} \subseteq X : \exists x \in X \text{ with } G\cdot x = \mathcal{O} \}$

(In other words, we need to pick orbit representatives to count ${}_G\backslash^X$.

Lemma: Assume X and G are finite. Then:

$$\frac{|X|}{|G|} = \sum_{\substack{\mathcal{O} \in {}_G\backslash^X \\ \overset{\shortparallel}{G\cdot x}}} \frac{1}{|\mathrm{Stab}_G(x)|}$$

Note: $|\mathrm{Stab}_G(x)| = |\mathrm{Stab}_G(\sigma \cdot x)|$ by Theorem (2), so the stabilizers of elements of $\mathcal{O}$ are the same.

Proof: We write X as a disjoint union of G-orbits

$$\Rightarrow \quad |X| = \sum_{\mathcal{O} \in {}_G\backslash^X} |\mathcal{O}|$$

If $\mathcal{O} = G\cdot x$, then $|\mathcal{O}| = |G\cdot x| \overset{\text{Theorem (1)}}{=} \left| {}^{G}/_{\mathrm{Stab}_G(x)} \right| = \dfrac{|G|}{|\mathrm{Stab}_G(x)|}$ $\square$

Example: $S_n \curvearrowright \{1, \ldots, n\}$ There is only one orbit.

Pick $n \in X$. Then $\mathrm{Stab}_{S_n}(n) \cong S_{n-1}$ $(\leq S_n)$

Lemma says $n = \dfrac{|S_n|}{|S_{n-1}|} = \dfrac{n!}{(n-1)!}$ ✓

Theorem (Burnside): $$\left| {}_G\backslash^X \right| = \frac{1}{|G|} \sum_{g \in G} |X^g|$$

Proof: We count orbits in two ways, by looking at the following subset $F \subseteq G \times X$

$$F := \{(g,x) \in G \times X \mid g \cdot x = x\} \quad (\text{ie } g \in \mathrm{Stab}_G(x))$$

(1) Fix $g \in G$ and count $(g,x) \in F$

$$|F| = \sum_{g \in G} |X^g|$$

$\hookrightarrow \#\{x \in X : g \cdot x = x\}$

(2) Fix $x \in G$ and count $(g,x) \in F$

$$|F| = \sum_{x \in X} |\mathrm{Stab}_G(x)| \underset{\downarrow}{=} \sum_{\mathcal{O} \in {}_G\backslash X} \underbrace{|\mathcal{O}|\,|\mathrm{Stab}_G(x)|}_{|G| \text{ by Theorem (1)}} \quad \text{for } \mathcal{O} = G \cdot x$$

$\mathrm{Stab}_G(x) = \mathrm{Stab}_G(\sigma \cdot x)$

$$= \sum_{\mathcal{O} \in {}_G\backslash X} |G| = |G|\,|{}_G\backslash X|$$

Comparing (1) and (2) gives $|{}_G\backslash X| = \frac{1}{|G|} \sum_{g \in G} |X^g|$