Lecture XIII: More examples of group actions

Recall: Some counting tricks we've learned for $G \circlearrowright X$

① $G/\mathrm{Stab}_G(x) \xrightarrow{\sim} G.x$ bijection $\forall x \in X$

$g.\mathrm{Stab}_G(x) \longmapsto g.x$

② $\mathrm{Stab}_G(x) \xrightarrow{\sim} \mathrm{Stab}_G(\sigma.x)$ group iso $\forall x \in X$ $\forall \sigma \in G$

$g \longmapsto \sigma g \sigma^{-1}$

③ $X^g \xrightarrow{\sim} X^{\sigma g \sigma^{-1}}$ bijection $\forall \sigma, g \in G$

$x \longmapsto \sigma.x$

Lemma: $$\frac{|X|}{|G|} = \sum_{\mathcal{O}: G.x \in G\backslash X} \frac{1}{|\mathrm{Stab}_G(x)|}$$

Burnside's Theorem: $$|G\backslash X| = \frac{1}{|G|} \sum_{g \in G} |X^g|$$

§13.1 Some special actions:

We have 3 adjectives for group actions $G \circlearrowright X$

Definition: We say a G-action on X is free if $\forall x \in X : (g.x = x \implies g = e)$

Equivalently, $\mathrm{Stab}_G(x) = \{e\}$ $\forall x \in X$

Note: By Theorem (1) $|G.x| = |G|$ $\forall x$, ie all orbits have the same size when the action is free. Moreover, by Lemma, $\frac{|X|}{|G|} = |G\backslash X|$

Definition: We say a G-action on X is transitive if $\forall x, y \in X$ $\exists g \in G$ such that $g.x = y$

Equivalently, $G.x = X$ $\forall x \in G$. In other words, there is only one orbit.

Note: By Lemma, $\frac{|X|}{|G|} = \frac{1}{|\mathrm{Stab}_G(x)|}$ for any $x \in G$. By Burnside: $1 = \frac{1}{|G|}\sum_{g \in G} |X^g|$.

Definition: We say a G-action on X is faithful if $G \xrightarrow{\tau} \mathrm{Aut}_{Set}(X)$ is 1-to-1.

Meaning: $g.x = x$ $\forall x \in X \implies g = e$ $\left[\bigcap_{x \in X} \mathrm{Stab}_G(x) = \{e\}\right]$

Note Free $\Rightarrow$ Faithful, but Faithful $\not\Rightarrow$ Free [see examples below]

Examples from Lecture 10:

(1) $D_{2n} \circlearrowright \mathbb{R}^2 \setminus \{(0,0)\}$
- Faithful ✓
- Free ✗ $\left(\mathrm{Stab}_G\left(\left[\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right]\right) = \langle s \rangle \ \forall x \in \mathbb{R}\right)$
  ($\exists$ orbits of size $n \neq |D_{2n}|$)
- Transitive ✗ (there are many orbits, all of finite size)

(2) $S_n \circlearrowright \{1,2,3,\ldots,n\}$, $n \geq 3$
- Faithful ✓
- Free ✗
- Transitive ✓

$\Big\}$ There is only one orbit of size $n < |S_n|$ for $n \geq 3$

§13.2 Some standard group actions

EXAMPLE 1: $X = G$ G acts as multiplication on the left

$$\begin{array}{ccc} G \times G & \xrightarrow{\alpha} & G \\ (g,x) & \longmapsto & gx \end{array} \quad \text{or} \quad \begin{array}{ccc} G & \longrightarrow & \mathrm{Aut}_{Set}(G) \\ g & \longmapsto & (\sigma(g) : h \longmapsto gh) \end{array}$$

⚠ $\sigma(g)$ is just a bijection on G, **NOT** a group homomorphism.

$\mathrm{Stab}_G(h) = \{e\}$, $G \cdot h = G$ $\forall h \in G$, $X^g = \begin{cases} \emptyset & g \neq e \\ G & g = e \end{cases}$

$\Rightarrow$ The action is free, transitive and faithful

Note: To get a right action, we take multiplication on the right.

EXAMPLE 2: $X = G/H$ = set of left cosets

$G \circlearrowright G/H$ by $g \cdot (g'H) := (gg')H$

- $\mathrm{Stab}_G(gH) = \{\sigma \in G \quad \sigma \cdot gH = (\sigma g)H = gH\}$

Note: $(\sigma g)H = gH \iff g^{-1}\sigma g \in H \iff \sigma \in gHg^{-1}$

$\Rightarrow \mathrm{Stab}_G(gH) = gHg^{-1}$. $\neq \{e\}$ if $H \neq \{e\}$, so not free if $H \neq \{e\}$.

- $G \circlearrowright G/H$ transitively (only one orbit $= G/H$)
- $\forall \sigma \in G$ $(G/H)^\sigma = \{gH : \sigma \cdot gH = gH\} = \{gH : \sigma \in gHg^{-1}\}$

EXAMPLE 3: $X = G$ $\quad G$ acts by conjugation

$G \circlearrowright G \qquad \sigma \cdot g = \mathrm{Conj}(\sigma)(g) = \sigma g \sigma^{-1}$

Recall: $\mathrm{Conj}: G \longrightarrow \mathrm{Aut}_{Gp}(G) \subseteq \mathrm{Aut}_{set}(G)$

Definition: Orbits of G under conjugation are called conjugacy classes.

- $\mathrm{Stab}_G(x) = \{g : gxg^{-1} = x\} =: Z_G(x)$ $\quad (\{g : gx = xg\} = Z_G(x))$

$\Rightarrow G \circlearrowright G$ by conjugation is free iff $Z_G(x) = \{e\}$ $\forall x \in G$.

Definition: $Z_G(x)$ is called centralizer of x. (elements that commute with x)

$\Rightarrow \quad G/Z_G(x) \xrightarrow{\sim}$ Conjugacy class of x $\quad$ is a bijection.

- Fixed points under conjugation $X^g = \{x \in G : gxg^{-1} = x\} = \{x \in G : x^{-1} g x = g\}$
$= Z_G(g)$

Remark: Kernel of $\mathrm{Conj}: G \longrightarrow \mathrm{Aut}_{Gp}(G) \subseteq \mathrm{Aut}_{set}(G)$

$$\begin{aligned}
\mathrm{Ker}(\mathrm{Conj}) &= \{g \in G : \mathrm{Conj}(g) = id_G\} \\
&= \{g \in G : ghg^{-1} = h \ \forall h\} \\
&= \{g \in G : gh = hg \ \forall h\} = Z(G)
\end{aligned}$$

Definition: $Z(G)$ is called the center of G.

$\Rightarrow G \circlearrowright G$ by conjugation is faithful iff $Z(G) = \{e\}$.

Note: Conj is a group homomorphism (Example §9.2) & $Z(G) = \mathrm{Ker}(\mathrm{Conj}) \subseteq G$

$\Rightarrow Z(G) \triangleleft G$.

§13.3 Counting conjugacy classes:

Fix G a finite group & let $G \circlearrowright G$ act by conjugation

- $\left| G/\mathrm{Stab}_G(x) \right| = |G \cdot x| \quad \Rightarrow \quad |C| = \dfrac{|G|}{|Z_G(\sigma_c)|}$ $\quad$ for $C = G \cdot \sigma_c$ conjugacy class.

- Counting Lemma says $1 = \dfrac{|X|}{|G|} = \displaystyle\sum_{C \text{ conj class}} \frac{1}{|Z_G(\sigma_c)|}$ $\qquad C = G \cdot \sigma_c$, σ_c a choice

$$\Longrightarrow \quad 1 = \sum_{C \text{ conj class}} \frac{|C|}{|G|} \qquad \left(\Longleftrightarrow \quad |G| = \sum_{\substack{C \text{ conj} \\ \text{class}}} |C| \quad ✓ \right)$$

- Burnside's Theorem: $\#$ of Conjugacy classes $= \frac{1}{|G|} \sum_{g \in G} |X^g| = \frac{1}{|G|} \sum_{g \in G} |Z_G(g)|$

Note: This is sometimes called the "Class Equation".

- Next, we discuss some fun identities for S_n.

Take $G = S_n \circlearrowright X = S_n$ by conjugation

Q: What are conjugacy classes?

A: Recall $\sigma(x_1 x_2 \cdots x_\ell)\sigma^{-1} = (\sigma(x_1)\ \sigma(x_2) \cdots \sigma(x_\ell))$

So Conjugacy Classes in $S_n \xleftrightarrow{1\text{-}1}$ cycle types (ie partitions of n)

Usually, people label conjugacy classes as either

(1) $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r)$ with $\lambda_1 + \cdots + \lambda_r = n$ (ordered decreasingly)

(2) or $\lambda = (\underbrace{1 \cdots 1}_{\ell_1 \text{ times}}, \underbrace{2 \cdots 2}_{\ell_2 \text{ times}}, \cdots\cdots)$ abbreviated as $(1^{\ell_1}, 2^{\ell_2}, 3^{\ell_3}, \ldots)$

In particular, $(1^{\ell_1}, 2^{\ell_2}, \ldots, n^{\ell_n})$ is a valid cycle type for a permutation $\sigma \in S_n$

$$\Longleftrightarrow \quad 1\cdot\ell_1 + 2\ell_2 + \cdots + n\ell_n = n.$$

Q: How many elements in S_n are there of a given cycle type?

A: Cycle Type $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots)$ or $(1^{\ell_1}, 2^{\ell_2}, 3^{\ell_3}, \ldots)$

$$C_\lambda = \{\sigma \in S_n \mid \sigma \text{ is a cycle of type } \lambda\}$$

$$|C_\lambda| = \frac{n!}{\lambda_1 \lambda_2 \cdots \lambda_r \ \ell_1! \ell_2! \cdots \ell_n!} = \frac{n!}{1^{\ell_1} \cdot 2^{\ell_2} \cdots (\ell_1! \ell_2! \cdots \ell_n!)}$$

↳ λ_1 ways of writing a cycle of length λ_1.

↳ reorder products of disjoint cycles of length λ_i.

§ 13.4 More examples:

EXAMPLE 4: Pick X set and G a group left acting on X

- Define: $\mathbb{C}[X] := \{ f : X \longrightarrow \mathbb{C} \text{ functions} \}$

This is a $\mathbb{C}$-vector space under pointwise addition and scalar multiplication
$\left((f_1+f_2)_{(x)} = f_{1(x)} + f_{2(x)} \;\&\; (\alpha * f)_{(x)} = \alpha f_{(x)};\ \forall x \in X,\ \forall \alpha \in \mathbb{C},\ \forall f, f_1, f_2 \in \mathbb{C}[X] \right)$

- $G \circlearrowright \mathbb{C}[X]$ via $(g \cdot f)_{(x)} = f(g^{-1} \cdot x)$

Why do we need the inverse? Because of 2^{nd} axiom of group actions on the left.

$$(g_1 \cdot (g_2 \cdot f))_{(x)} = (g_2 \cdot f)(g_1^{-1} \cdot x) = f(g_2^{-1} \cdot (g_1^{-1} \cdot x)) = f((g_2^{-1} g_1^{-1}) \cdot x)$$
$$= f((g_1 g_2)^{-1} \cdot x) = ((g_1 g_2) \cdot f)_{(x)} \qquad \forall x$$

<u>Note</u>: If $X \circlearrowleft G$, then $\mathbb{C}[X] \circlearrowleft G$ with $(f \cdot g_2)_{(x)} = f(x \cdot g_2)$ $\forall x$

- Same ideas work with $\mathbb{R}[X] := \{ f : X \longrightarrow \mathbb{R} \text{ functions} \}$

<u>Application</u>: An alternative proof of Burnside's Theorem (optional)

Here is an alternative proof of Burnside's Theorem that uses a bit of Linear Algebra

<u>IDEA</u>: Change $|X|$ to the dimension of a real vector space ($\subseteq \mathbb{R}[X]$ vector space)

$\mathbb{R}[X]$ is a vector space with basis $B = \{ e_x : x \in X \}$. Here, $e_x(y) = \begin{cases} 1 & \text{if } y = x \\ 0 & \text{else} \end{cases}$

$\Rightarrow \dim(\mathbb{R}[X]) = |X|$. Set $N = |X|$.

- Define $\text{Functions}({}_G\backslash^X, \mathbb{R}) := \{ f : X \longrightarrow \mathbb{R} \quad f(g \cdot x) = f(x) \ \forall g \in G\ \forall x \in X \}$
(ie, functions with constant values along orbits of the action $G \circlearrowright X$)
- $\text{Functions}({}_G\backslash^X, \mathbb{R}) \leq \mathbb{R}[X]$ vector subspace.

Basis for $\text{Functions}({}_G\backslash^X, \mathbb{R}) = \{ \sum_{x \in \mathcal{O}} e_x \}_{\mathcal{O} \in {}_G\backslash^X}$ so its dimension is $|{}_G\backslash^X|$.

We define a map $\rho : G \longrightarrow GL_N(\mathbb{R})$ $\quad \rho(g) : e_x \longmapsto e_{g \cdot x}$

[ie (x,y)-entry of $[\rho(g)]_{BB} = \begin{cases} 1 & \text{if } x = g \cdot y \\ 0 & \text{else} \end{cases}$] $\Rightarrow$ $\boxed{\rho(g)(f)_{(x)} = g \cdot f\,(x)}$ check it on $\{e_y\}_y$

Claim: Trace of $\rho(g) = |X^g|$ $\quad$ ($\#\{x : x = g\cdot x\}$)

$\Rightarrow$ Trace of $\left(\frac{1}{|G|} \sum_{g\in G} \rho(g) \right) = \frac{1}{|G|} \sum_{g\in G}$ Trace of $\rho(g) = \frac{1}{|G|} \sum_{g\in G} |X^g|$.

↪ averaging operator

- Burnside's Identity requires us to prove the following identity:

$$\text{dimension Functions}(G\backslash X, \mathbb{R}) = \text{Trace of} \left(\frac{1}{|G|} \sum_{g\in G} \rho(g) \right) \qquad (*)$$

Now we have an inclusion of vector spaces and a map

$$\begin{tikzcd} V_1 := \text{Functions}(G\backslash X, \mathbb{R}) \arrow[r, hook, "i"'] & \text{Functions}(X, \mathbb{R}) =: V \arrow[l, bend right, "Av"'] \end{tikzcd}$$

We define the averaging operator $Av : V \longrightarrow V_1$ as $Av(f) = \frac{1}{|G|} \sum_{g\in G} g\cdot f$.

Thus, $Av(f)(x) = \frac{1}{|G|} \sum_{g\in G} (g\cdot f)(x) = \frac{1}{|G|} \sum_{g\in G} f(g^{-1}\cdot x) = \frac{1}{|G|} \sum_{g\in G} \rho(g)(f)(x)$

$= \left(\frac{1}{|G|} \sum_{g\in G} \rho(g) \right)(x)$

Note: $Av(f) \in V_1$ because we have that $Av(f)(h\cdot x) = Av(f)(x)$ $\forall h\in G$ $\forall x \in X$.

$$Av(f)(h\cdot x) = \frac{1}{|G|} \sum_{g\in G} f(g\cdot(h\cdot x)) = \frac{1}{|G|} \sum_{g\in G} f((gh)\cdot x) \underset{\text{relabel } w = gh}{=} \frac{1}{|G|} \sum_{w\in G} f(w\cdot x) = Av(f)(x)$$

- Claim 1: $Av \circ i(f) = f$ $\quad \forall f \in V_1$

Pf/ By direct computation:

$$Av(i(f))(x) = \frac{1}{|G|} \sum_{g\in G} (i(f))(g\cdot x) = \frac{1}{|G|} \sum_{g\in G} \underbrace{f(g\cdot x)}_{\substack{\| \\ f(x) \ (f\in V_1)}} = \frac{f(x)|G|}{|G|} = f(x) \quad \forall x \in X$$

$\square$

- Consider $P := i \circ Av : V \longrightarrow V$

Claim 2: $P^2 = P$ (it's a projection), $\operatorname{Im} P \cong V_1$, $\operatorname{Ker} P = \operatorname{Ker} Av$

Pf/ $(i\circ Av)\circ(i \circ Av) = i\circ(Av\circ i)\circ Av \underset{\hookrightarrow \text{ Claim 1}}{=} i \circ id_{V_1} \circ Av = i \circ Av$

$\operatorname{Im} P \subseteq V_1 \subseteq V$ $\quad$ & $\quad$ $i(f) = P(i(f))$ for $f \in V_1$. by Claim 1

$\operatorname{Ker}(P) = \{ f\in V : i\circ Av(f) = 0\} = \{ f \in V : Av(f) = 0\} = \operatorname{Ker}(Av)$ because i is injective

$\Rightarrow$ Consider B_1 a basis for V_1 & B_2 a basis for

Since P is a projection, we have $\mathrm{Ker}(A_v) \oplus \underbrace{\mathrm{Im}(P)}_{=V_1} = V$.

Then
$$[P]_{B_1 \cup B_2} = \begin{array}{cc} \begin{array}{cc} |B_1| & |B_2| \end{array} & \\ \left[\begin{array}{c|c} \mathrm{Id} & 0 \\ \hline 0 & 0 \end{array}\right] & \begin{array}{c} |B_1| \\ |B_2| \end{array} \end{array}$$

<u>Conclusion:</u> $\mathrm{Trace}(A_v) \underset{\substack{\downarrow \\ i \text{ inclusion}}}{=} \mathrm{Trace}(i \circ A_v) = \mathrm{Trace}(P) = |B_1| = \dim V_1$. which is exactly the equality $(*)$ we were seeking.