Lecture XV: Applications of Sylow Theorems

Recall: G finite group $|G| = p^r m$ $p \nmid m$. Then, $H \leq G$ is Sylow p-subgroup if $p^r = |H|$

$\mathrm{Syl}_p(G) = \{ P \leq G : P \text{ is a Sylow } p\text{-subgroup of } G\}$

Sylow Theorems: G finite group, $p \mid |G|$ prime. Then,

(1) Sylow p-groups exist

(2) Sylow p-subgroups are unique up to conjugation

(3) $n_p = \#$ Sylow p-subgroups of G $\quad n_p \equiv 1 \mod (p)$ & $n_p \mid |G|$.

Lemma: If $G \circlearrowright X$ and $|G| = p^r$ (a power of a prime), then $|X| \equiv |X^G| \mod p$

Here, $X^G := \{ x \in X : g \cdot x = x \ \forall g \in G\}$ $\quad \left(= \bigcap_{g \in G} X^g\right)$

§15.1 Proof of Sylow Theorems:

Proof of (1): (Lecture 14). $\quad X = \{p^r\text{-element subsets of } G\}$ $\quad |X| = \binom{p^r m}{p^r} \equiv m \pmod p$

$G \circlearrowright X$ $\qquad gH = \{gh_1, \ldots, gh_{p^r}\}$ for $H = \{h_1, \ldots, h_{p^r}\} \in X$ & $g \in G$.

$\Rightarrow \mathrm{Stab}_G(H) \subseteq \{ h_i^{-1} h_1 : i = 1, \ldots, p^r\}$ has size $\leq p^r$.

Since $p \nmid m \Rightarrow p \nmid |X| = \sum_{\mathcal{O} \in G\backslash X} |\mathcal{O}|$ $\quad$ so $\exists\, \mathcal{O} \in G\backslash X$ with $p \nmid |\mathcal{O}|$

Since $|\mathcal{O}| \mid |G|$, we get $|\mathcal{O}| \mid m$. Write $\mathcal{O} = G \cdot H$ for some H.

Conclude: $|\mathrm{Stab}_G(H)| = p^r$ because $\underset{p^r m}{\underset{\parallel}{|G|}} = \underset{\text{coprime to } p}{\underset{\wr}{|\mathcal{O}|}}\, |\mathrm{Stab}_G(H)|$, So $\mathrm{Stab}_G(H) \in \mathrm{Syl}_p(G)$. $\square$

Proof of (2): Will be a direct consequence of the following more general result:
(taking H to be any Sylow p-subgroup of G)

Theorem: Let P be a Sylow p-subgroup of G and $H \leq G$ be a p-group.
Then, $\exists\, g \in G$ such that $H \subseteq gPg^{-1}$.

Proof: We first note that $H \subseteq gPg^{-1} \iff Hg \subseteq gP \iff HgP \subseteq gP$,
meaning the coset $gP \in G/P$ is fixed by H where $G \circlearrowright G/P$ by left multiplication. This is what we will prove.

- We consider $H \circlearrowright G/P$ via $h\cdot(g'P) = hg'P$ for each $g'P \in G/P$.

Now: $|H| = p^s$ ie $s \le r$ and G/P has m elements with $m \not\equiv 0 \mod (p)$

By Lemma §12.1 $m = |G/P| \equiv \#$ H-fixed points of G/P. mod (p)

Since $m \not\equiv 0 \mod (p)$, we have at least 1 H-fixed point of G/P, ie $\exists\, g \in G$ s.t

$h(gP) = gP \quad \forall h \in H$, ie $H \le gPg^{-1}$. $\square$

<u>Proof of (3)</u>: Let $\mathcal{S}_p$ = set of Sylow p-subgroups of G

So far, we know:
- $\mathcal{S}_p \neq \emptyset$ (by (1))
- $G \circlearrowright \mathcal{S}_p$ by conjugation is Transitive. (by (2))

We write $n_p = \# \mathcal{S}_p$.

As $\mathcal{S}_p$ has a single orbit, we know $n_p = \dfrac{|G|}{|\mathrm{Stab}_G(P_0)|} \mid |G|$ for any $P_0 \in \mathcal{S}_p$

ie $n_p \mid p^r m$

- We show $n_p \equiv 1 \mod p$ using Lemma §12.1. $(\Rightarrow n_p \mid m)$

Choose $P_0 \in \mathcal{S}_p$ and let $\underset{\sigma}{\overset{}{P_0}} \circlearrowright \underset{P}{\mathcal{S}_p}$ induced by $G \circlearrowright \mathcal{S}_p$ conjugation action

$$\sigma \in P_0, \quad P \in \mathcal{S}_p \quad (\sigma \cdot P = \sigma P \sigma^{-1})$$

<u>Note</u>: The action of P_0 need <u>not</u> be transitive

By the Lemma: $n_p = |\mathcal{S}_p| \equiv \underbrace{|\mathcal{S}_p^{P_0}|}_{= \#\, P_0\text{-fixed points of } \mathcal{S}_p} (\mathrm{mod}\ p)$

<u>Claim</u>: $\{P_0\text{-fixed points of } \mathcal{S}_p\} = \{P_0\}$

Pf/ $\cdot(\supseteq)$ P_0 is fixed by P_0, so $(\supseteq)$ holds.

$(\subseteq)$ Let $Q \in \{P_0\text{-fixed points of } \mathcal{S}_p\}$, so $Q \le G$ is a Sylow p-subgroup of G

and $xQx^{-1} = Q \quad \forall x \in P_0$.

Consider $H := \{g \in G : gQg^{-1} = Q\} \le G$

$$H = \mathrm{Stab}_G(Q) \text{ under } G \circlearrowright \mathcal{S}_p \text{ by conjugation.}$$

$\Rightarrow$ $P_0 \le H$ & $Q \le H$ are 2 Sylow p-subgroups of H ($p^r \le |H|$ & $|H| \mid p^r m$ with $p \nmid m$)

By Sylow Theorem (2) $\exists\, h \in H$ st $\underbrace{hQh^{-1}}_{= Q \text{ as } h \in H} = P_0$

We conclude that $Q = P_0$. $\square$

Corollary: If $P_0 \leq G$ is the only Sylow p-subgroup of G, then $P_0 \trianglelefteq G$

Proof: gP_0g^{-1} is a p-Sylow subgroup of G $\forall g \in G$. By uniqueness, we have $gP_0g^{-1} = P_0$ $\forall g$. So, $P_0 \trianglelefteq G$ by definition.

NEXT GOAL: Use Sylow Theorems for solving 2 types of problems:

(I) Given a finite group G and a prime p with $p \mid |G|$, determine $\mathrm{Syl}_p(G)$

(II) Find one non-trivial $(\neq \{e\})$, proper $(\neq G)$ normal subgroup of G

⚠ These questions may not be solvable via Sylow Theorems, but for many G's they are.

§15.2 Sylow subgroups of S_p:

Set $G = S_p$ with p prime. Then, $|G| = p! = (p-1)!\,p$

$\Rightarrow$ Any Sylow p-subgroup P has size p. so $P \simeq \mathbb{Z}/p\mathbb{Z}$.

- P has $p-1$ elements of order p
- Each element of order p generates a Sylow p-subgroup
- Elements of order p in S_p are p-cycles
- If $\mathrm{Syl}_p(S_p) = \{P_1, \ldots, P_{n_p}\}$, then $P_i \cap P_j = \{id\}$ if $i \neq j$

$\Rightarrow$ # elements of order p $= (p-1)\,n_p$.

Proposition: $n_p(S_p) = (p-2)!$

Proof: $\dfrac{\#\ p\text{-cycles}}{p-1} = \dfrac{p!/p}{p-1} = (p-2)!$

§15.3 Sylow subgroups of S_4:

$G = S_4$ $\qquad |G| = 24 = 2^3 \cdot 3$

Q: What about $\mathrm{Syl}_3(S_4)$? $\qquad n_3 \equiv 1\ (3)$ & $n_2 \mid 8$ $\Rightarrow$ $n_2 = 1$ or 4.

$P \in \mathrm{Syl}_3(S_4)$ has order 3, so it's cyclic, generated by a 3-cycle.

\# 3-cycles in S_4 $= \binom{4}{3} \frac{3!}{3} = 4 \cdot 2 = 8$

Each P has 2 elements of order 3

$P_i, P_j \in \mathrm{Syl}_3(P_4)$ with $i \neq j$ has $P_i \cap P_j = \{e\}$ because $|P_i| = |P_j| = 3$ is a prime number

$\Rightarrow$ $n_3 \cdot 2$ = Total number of 3-cycles gives $n_3 = \frac{8}{2} = 4$.

Conclude: $\mathrm{Syl}_3(S_4) = \{ \langle (123) \rangle, \langle (124) \rangle, \langle (134) \rangle, \langle (234) \rangle \}$

We'll discuss $\mathrm{Syl}_2(S_4)$ tomorrow.