Lecture XIX: Direct Products I

Recall: Last Time, we used Sylow Theorems to show

G gp, $|G| = 45 \implies G \simeq \begin{pmatrix}\text{A group of} \\ \text{size } 9\end{pmatrix} \times \begin{pmatrix}\text{A group of} \\ \text{size } 5\end{pmatrix}$

This is an example of a direct product

Lemma: Given G_1, G_2 groups, the cartesian product $G_1 \times G_2$ is a group with

- $e_{G_1 \times G_2} = (e_{G_1}, e_{G_2})$
- $* : G \longrightarrow G \qquad (g_1, g_2) * (g_1', g_2') = (g_1 *_1 g_1', g_2 * g_2')$.

Proof: Associativity and existence of inverses are inherited from both G_1 & G_2

Indeed: $(g_1, g_2)^{-1} = (g_1^{-1}, g_2^{-1})$. $\square$

§19.1 Direct Products:

Recall the following lemma from §18.1

Lemma: Let G be a group, and N_1, N_2 be 2 subgroups of G. Assume that

(1) $N_1, N_2 \trianglelefteq G$

(2) $N_1 \cap N_2 = \{e\}$.

Then, $ab = ba \quad \forall a \in N_1,\ b \in N_2$.

• **Lemma 2:** Let G be a group and $N_1, N_2 \trianglelefteq G$ normal subgroups s.t:

(1) G is generated by $N_1 \cup N_2$

(2) $N_1 \cap N_2 = \{e\}$

Then $f: N_1 \times N_2 \longrightarrow G$ is a group isomorphism.
$\qquad (x_1, x_2) \longmapsto x_1 x_2$

Proof: Since $N_1, N_2 \trianglelefteq G$ and $N_1, N_2 = \{e\}$, Lemma ensures that f is a group homomorphism:

$$f((x_1, x_2) * (x_1', x_2')) = f((x_1 x_1', x_2 x_2')) = (x_1 x_1')(x_2 x_2') = x_1 (x_1' x_2) x_2'$$

$$= x_1 (x_2 x_1') x_2' = (x_1 x_2)(x_1' x_2') = f(x_1, x_2) * f(x_1', x_2')$$

$\left.\begin{matrix} x_2 \in N_2 \\ x_1' \in N_1 \end{matrix}\right\} \Rightarrow x_1' x_2 = x_2 x_1'$

• f is surjective: Since $n_1 n_2 = n_2 n_1 \ \forall n_1 \in N_1, \forall n_2 \in N_2$, any word on N_1 & N_2

can be written as xy with $x \in N$ and $y \in N_2$ $\Rightarrow$ Any $g \in G$ lies in $\mathrm{Im}(f)$.

(Proof by complete induction on the length of a word

Write $w = a_1 a_2 \cdots a_r$ $\quad$ $a_1 \in N_1, a_3 \in N_1, \ldots$ $a_2 \in N_2, a_4 \in N_2$ $\quad$ (allow $a_1 = e$) $\quad$ length$(w) = r$.

- Base cases:
  - $w = a$ with $a \in N_1$ $\Rightarrow$ $w = \underset{(a \in N_1)}{a \cdot e}$
  - $w = a_1 a_2$ $\quad$ $a_1 \in N_1, a_2 \in N_2$ ✓

- Inductive step: $w = a_1 a_2 \cdots a_{r-2} a_{r-1} a_r = a_1 a_2 \cdots \underbrace{(a_{r-3} a_{r-1})}_{\in \text{ same subgroup}} \underbrace{(a_{r-2} a_r)}_{\in \text{ same subgroup}}$

  $a_{r-2} a_{r-1} = a_{r-1} a_{r-2}$

We have reduced the length of w by 2., so by the inductive hypothesis $w = a_1' a_2'$ with $a_1' \in N_1$ & $a_2' \in N_2$).

- f is injective $\quad f((x_1, x_2)) = e \Leftrightarrow x_1 x_2 = e \Leftrightarrow \underset{\in N_1}{x_1} = \underset{\in N_2}{x_2^{-1}}$

$\underset{N_1 \cap N_2 = \{e\}}{\Rightarrow}$ $x_1 = x_2^{-1} = e$, i.e. $x_1 = x_2 = e$. $\Rightarrow (x_1, x_2) = e_{G_1 \times G_2}$.

$\square$

Remark: This argument can be easily generalized to a finite number of normal subgroups.

Proposition: Let G be a group and $N_1, \ldots, N_k \trianglelefteq G$ normal subgroups s.t.:

(1) G is generated by $\bigcup_{i=1}^{k} N_i$

(2) $N_i \cap (N_1 \cdots N_{i-1} N_{i+1} \cdots N_k) = \{e\}$ $\quad \forall i$.

Then $f : N_1 \times N_2 \times \cdots \times N_k \longrightarrow G$ is a group isomorphism.

$(x_1, x_2, \ldots, x_k) \longmapsto x_1 x_2 \cdots x_k$

Proof: Note that $N_j \leq (N_1 \cdots N_{i-1} N_{i+1} \cdots N_k)$ $\forall j \neq i$ since $n_j = e.n_j e$ $\forall j$

Thus, by (2), $N_i \cap N_j = \{e\}$ $\forall i \neq j$. So, the Lemma ensures

(*) $\quad x_i x_j = x_j x_i$ if $x_i \in N_i, x_j \in N_j$ & $i \neq j$.

- We define the product structure on $N_1 \times \cdots \times N_k$ componentwise.
- By (*) f is a group homomorphism.
- Condition (*) $\Rightarrow$ f is surjective
- f injective: It's enough to check the following

Claim: $x_{i_1} \cdots x_{i_j} = e$ with $i_1 \leq \cdots \leq i_j$ & $x_{i_s} \in N_{i_s}$ $\forall s = 1, \ldots, j$ $\Rightarrow x_{i_s} = e$ $\forall s$.

Pf/ By induction on $2 \leq j \leq k$

Base case: $k=2$ is True by Lemma 2

Inductive Step: Assume the statement is true for $s < j \leq k$.

$$x_{i_1} \cdots x_{i_s} x_{i_{s+1}} = e \iff x_{i_1} = x_{i_{s+1}}^{-1} \cdots x_{i_2}^{-1} \underset{x_{i_s}^{-1}\text{'s commute}}{=} x_{i_2}^{-1} \cdots x_{i_{s+1}}^{-1} \in N_{i_2} \cdots N_{i_{s+1}} \subseteq N_1 \cdots N_{i_1-1} N_{i_1+1} \cdots N_k$$

$\Rightarrow x_{i_1} \in N_{i_1} \cap N_1 \cdots N_{i_1-1} N_{i_1+1} \cdots N_k = \{e\}$, so $x_{i_1} = e$ & $\underbrace{x_{i_2} \cdots x_{i_{s+1}}}_{s \text{ terms}} = e$.

By inductive hypotheses, $x_{i_2} = \cdots = x_{i_{s+1}} = e$.

The claim follows from this. □

§19.2 Classification of finite abelian groups:

As a Corollary, we can classify all finite abelian groups. Here is the first step

Theorem 1: Let G be a finite abelian group. If $|G| = n$ is written into its prime factors $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ (p_i all distinct primes, $a_i \in \mathbb{Z}$ $\forall i$) then $\exists$ $P_i \trianglelefteq G$ of order $p_i^{a_i}$ such that $G \simeq P_1 \times \cdots \times P_k$. Furthermore, this decomposition is unique.

Proof: First, we prove the existence of such a decomposition.

Let $P_i \in Syl_{p_i}(G)$. By construction, $|P_i| = p_i^{a_i}$ and $P_i \trianglelefteq G$ $\forall i$ because G is abelian.

We check $P_1, \ldots, P_k$ satisfy conditions (1) and (2)

We start with (2). $H_i := P_1 \cdots P_{i-1} P_{i+1} \cdots P_k \leq G$ $\forall i$ because G is abelian.

Now, $\varphi_i : P_1 \times \cdots \times P_{i-1} \times P_{i+1} \times \cdots \times P_k \twoheadrightarrow H_i \leq G$ is a group homomorphism,

so by the first iso theorem $H_i \simeq (P_1 \times \cdots \times P_{i-1} \times P_{i+1} \times \cdots \times P_k) / \ker \varphi_i$,

so $|H_i| \mid |P_1 \times \cdots \times P_{i-1} \times P_{i+1} \times \cdots \times P_k| = \dfrac{n}{p_i^{a_i}}$.

As a consequence $(|H_i|, |P_i|) = 1$ $\forall i$ $\Longrightarrow$ $P_i \cap H_i = \{e\}$ $\forall i$.

- Let $W = \langle P_1, \ldots, P_k \rangle \leq G$. Since $P_i \leq W$, we get $p_i^{a_i} \mid |W|$ $\forall i$

Thus $\operatorname{lcm}(p_1^{a_1}, \ldots, p_k^{a_k}) = n = |G| \mid |W|$, so $W = G$.

By Proposition, $P_1 \times \cdots \times P_k \simeq G$ via $(x_1, \ldots, x_k) \longmapsto x_1 \cdots x_k$. □

- **For uniqueness:** k is unique since $n = p_1^{a_1} \cdots a_k^{a_k}$.

  Assume $P_1 \times \cdots \times P_k \cong H_1 \times \cdots \times H_k$ for P_i, H_i groups of order $p_i^{a_i}$

  Note $P_i \leq P_1 \times \cdots \times P_k$ via $x \longmapsto (e, \ldots, e, \underset{i^{th}}{x}, e, \ldots, e)$

Since $|P_1 \times \cdots \times P_k| = n$, we see P_i is a Sylow p_i-subgroup of $P_1 \times \cdots \times P_k$.

Furthermore $Syl_{p_i}(P_1 \times \cdots \times P_k) = \{P_i\}$ since $P_1 \times \cdots \times P_k$ is abelian.

Similarly $Syl_{p_i}(H_1 \times \cdots \times H_k) = \{H_i\}$

For any isomorphism $\varphi: P_1 \times \cdots \times P_k \xrightarrow{\sim} H_1 \times \cdots \times H_k$, $|\varphi(P_i)| = |P_i| = p_i^{a_i}$

so $\varphi(P_i) \in Syl_{p_i}(H_1 \times \cdots \times H_k) = \{H_i\}$

Similarly, $\varphi^{-1}(H_i) \in Syl_{p_i}(P_1 \times \cdots \times P_k)$. Thus, φ induces an isomorphism

$H_i \simeq P_i$. $\forall i$. Uniqueness follows. $\square$

- From Theorem 1, we need only classifying the possibilities for $P_1 \cdots P_k$, ie we are reduced to classifying abelian p-groups. We'll do this next Time!