Lecture XXIII: Semi-direct Products II

Recall: A group G is a semi-direct product of subgroups H and N if
(1) $H \leq G$ and $N \trianglelefteq G$
(2) $HN = NH = G$ Write $G = N \rtimes H$
(3) $H \cap N = \{e\}$

Conjugation gives an action $G \curvearrowright N$, restricting it to H we get $H \curvearrowright N$ left-action, or equivalently, a group homomorphism

$$\alpha : H \longrightarrow \mathrm{Aut}_{Gp}(N)$$
$$h \longmapsto \mathrm{Conj}(h)$$

Today: We give a different construction of semi-direct products starting from such a group homomorphism α.

§23.1 Constructing Semi-direct Products:

Assume we are given two groups H and N, and a group homomorphism

$$\alpha : H \longrightarrow \mathrm{Aut}_{Gp}(N) := \{\varphi : N \to N : \varphi \text{ group isomorphism}\}$$

Conditions on α:
- $\alpha(h) : N \to N$ iso, so $\alpha(h)^{-1} = \alpha(h^{-1}) \quad \forall h$
- $\alpha(e_H) = id_N$
- $\alpha(h_1 *_H h_2) = \alpha(h_1) \circ \alpha(h_2) \quad : N \to N$

$$[\ \alpha(h_1 h_2)(n) = \alpha(h_1)(\alpha(h_2)(n)) \quad \forall n \in N \quad \forall h_1, h_2 \in H.\ ]$$

Using α we can define a new binary operation on $G = N \times H$, different than the coordinatewise one. More precisely:

$$\boxed{(n_1, h_1) *_\alpha (n_2, h_2) = (n_1\, \alpha(h_1)(n_2),\ h_1 h_2)} \quad \forall\ n_1, n_2 \in N,\ h_1, h_2 \in H$$

Lemma: The operation $*_\alpha$ defines a group structure on the set $G = N \times H$.

We denote G endowed with this operation by $N \rtimes_\alpha H$ (H acts on N via α, i.e. $h \bullet n = \alpha(h)(n)$) reserving the notation $N \times H$ for the coordinatewise operation.

Proof: We need to check 3 things:

(i) the operation defined is associative;

(ii) there is a neutral element ;
(iii) every element has an inverse .

(i) Set $g_1 = (n_1, h_1)$, $g_2 = (n_2, h_2)$, $g_3 = (n_3, h_3)$ $\quad n_1, n_2, n_3 \in N$, $h_1, h_2, h_3 \in H$

$$(g_1 *_\alpha g_2) *_\alpha g_3 = (n_1 \cdot \alpha(h_1)(n_2), h_1 h_2) *_\alpha (n_3, h_3)$$
$$= ((n_1 \alpha(h_1)(n_2)) \alpha(h_1 h_2)(n_3), (h_1 h_2) h_3)$$
$$g_1 *_\alpha (g_2 *_\alpha g_3) = (n_1, h_1) *_\alpha (n_2 \alpha(h_2)(n_3), h_2 h_3)$$
$$= (n_1 \alpha(h_1)(n_2 \alpha(h_2)(n_3)), h_1 (h_2 h_3))$$

We check the corresponding entries of these 2 elements of $N \times H$ agree.

- $(h_1 h_2) h_3 = h_1 (h_2 h_3)$ because the operation on H is associative
- $(n_1 \alpha(h_1)(n_2)) \alpha(h_1 h_2)(n_3) = (n_1 \alpha(h_1)(n_2))(\alpha(h_1) \circ \alpha(h_2))(n_3)$

$$= (n_1 \alpha(h_1)(n_2)) (\alpha(h_1)(\alpha(h_2)(n_3))) \underset{\text{Assoc on } N}{=} n_1 (\underbrace{\alpha(h_1)}(n_2) \underbrace{\alpha(h_1)}(\alpha(h_2)(n_3)))$$

$$\underset{\alpha(h_1) \text{ gp hom on } N}{=} n_1 \alpha(h_1)(n_2 \alpha(h_2)(n_3))$$

Conclude : $(g_1 *_\alpha g_2) *_\alpha g_3 = g_1 *_\alpha (g_2 *_\alpha g_3)$.

(ii) Claim : $e_G = (e_1, e_2)$ $\quad e_1 = e_N$, $e_2 = e_H$. identities on N and H.

$$(e_1, e_2) *_\alpha (n, h) = (e_1 \alpha(e_2)(n), e_2 h) \underset{\alpha(e_2) = id_N}{=} (e_1 \cdot n, e_2 h) = (n, h)$$

$$(n, h) *_\alpha (e_1, e_2) = (n \alpha(h)(e_1), h e_2) \underset{\alpha(h) \text{ iso on } N}{=} (n \cdot e_1, h e_2) = (n, h)$$

(iii) Claim : Inverse of $(n, h) = ((\alpha(h))^{-1}(n^{-1}), h^{-1}) = (\alpha(h^{-1})(n^{-1}), h^{-1})$

- $(n, h)(\alpha(h^{-1})(n^{-1}), h^{-1}) = (n \alpha(h)(\alpha(h^{-1})(n^{-1})), h h^{-1})$

$$\underset{\alpha \text{ gp hom}}{=} (n \underbrace{(\alpha(h) \circ \alpha(h^{-1}))}_{= \alpha(hh^{-1}) = \alpha(e_H) = id_N}(n^{-1}), e_H) = (n \, id_N(n^{-1}), e_H) = (n n^{-1}, e_H)$$
$$= (e_N, e_H) = e_G$$

- $(\alpha(h^{-1})(n^{-1}), h^{-1}) (n, h) = (\alpha(h^{-1})(n^{-1}) \alpha(h^{-1})(n), h^{-1} h)$

$$\underset{\alpha(h^{-1}) \text{ gp hom}}{=} (\alpha(h^{-1})(n^{-1} n), e_H) = (\alpha(h^{-1})(e_N), e_H) \underset{\alpha(h^{-1})(e_N) = e_N}{=} (e_N, e_N) = e_G$$

$\square$

Remark : If α is trivial ($\alpha(h) = id_N \ \forall h$), then $N \rtimes_\alpha H = N \times H$ ($*_\alpha$ = coordinatewise product)

Next, we show that H & N can be viewed as subgroups of $N \rtimes_\alpha H$, and N is normal in $N \rtimes_\alpha H$:

We consider $i_1 : H \longrightarrow N \rtimes_\alpha H$, $h \longmapsto (e_N, h)$ and $i_2 : N \longrightarrow N \rtimes_\alpha H$, $n \longmapsto (n, e_H)$

Proposition: The maps i_1, i_2 are injective group homomorphisms, allowing us to view H and N as subgroups of $G = N \rtimes_\alpha H$. Furthermore,

(1) $H \leq G$, $N \trianglelefteq G$

(2) $N \cdot H = G$

(3) $N \cap H = \{ e_G := (e_N, e_H) \}$

Proof: First, we check i_1 is a group homomorphism:

$$(e_2, h_1) *_\alpha (e_2, h_2) = (e_2\, \alpha(h_1)(e_2), h_1 h_2) \underset{\substack{\uparrow \\ \alpha(h)(e_2) = e_2 \quad (\alpha(h) \text{ gp hom})}}{=} (e_2 e_2, h_1 h_2) = (e_2, h_1 h_2)$$

So $i_1(h_1) *_\alpha i_1(h_2) = i_1(h_1 h_2) \quad \forall h_1, h_2 \in H$

- Next, we check i_2 is a group homomorphism:

$$(n_1, e_1) *_\alpha (n_2, e_1) = (n_1\, \alpha(e_1)(n_2), e_1 e_1) \underset{\substack{\uparrow \\ \alpha(e_1) = id_N}}{=} (n_1 \cdot n_2, e_1)$$

So $i_2(n_1) *_\alpha i_2(n_2) = i_2(n_1 n_2) \quad \forall n_1, n_2 \in N$

- By construction $e_G = (e_1, e_2)$, so $Ker(i_1) = \{e_1\}$, $Ker(i_2) = \{e_2\}$.

(1). $H \leq G$ ie $i_1(H) \leq G$

- Check $i_2(N) \trianglelefteq G$:

$$(n_1, h_1) *_\alpha (n_2, e_1) *_\alpha (n_1, h_1)^{-1} = (n_1, h_1) *_\alpha (n_2, e_1) *_\alpha (\alpha(h_1^{-1})(n_1^{-1}), h_1^{-1})$$

$$= (n_1\, \alpha(h_1)(n_2), h_1 e_1) *_\alpha (\alpha(h_1^{-1})(n_1^{-1}), h_1^{-1})$$

$$= (n_1\, \alpha(h_1)(n_2)\, \alpha(h_1)(\alpha(h_1^{-1})(n_1^{-1})), h_1 h_1^{-1})$$

$$= (n_1\, \alpha(h_1)(n_2)\, \underbrace{(\alpha(h_1) \circ \alpha(h_1^{-1}))}_{= \alpha(h_1 h_1^{-1}) = \alpha(e_1) = id_N}(n_1^{-1}), e_1) = (n_1\, \alpha(h_1)(n_2)\, n_1^{-1}, e_1)$$

$$= i_2(n_1\, \alpha(h_1)(n_2)\, n_1^{-1}) \qquad \forall n_1, n_2 \in N, \quad \forall h_1 \in H.$$

(2) $NH = \{ i_2(n) *_\alpha i_1(h) : n \in N, h \in H \}$

$$i_2(n) *_\alpha i_1(h) = (n, e_1) *_\alpha (e_2, h) = (n\, \alpha(e_1)(e_2), e_1 h) = (n, h)$$

$\Rightarrow N \cdot H = N \times H$ as sets.

(3) $N \cap H = \{e\}$ by definition of i_1 & i_2. $\square$

Corollary Given N, H groups & $\alpha: H \to \mathrm{Aut}_{Gp}(N)$ gp homomorphism, the group $N \rtimes_\alpha H$ is the semi-direct product of $N \triangleleft N \rtimes_\alpha H$ and $H \leq N \rtimes_\alpha H$.

Note: Proposition $\Rightarrow$ $NH = HN = N \rtimes_\alpha H$ by Lemma 22.1.

§23.2 Equivalence of two constructions:

Our next goal: show both constructions of semi-direct products (internal and external)

- Let $\mathcal{G}$ be a group which is a semi-direct product of two groups H & N

$$\left(\text{ie, } \begin{array}{l} H \leq \mathcal{G} \\ N \trianglelefteq \mathcal{G} \end{array}, \quad H \cdot N = N \cdot H = \mathcal{G}, \quad N \cap H = \{e\} \right)$$

Let $\alpha: H \longrightarrow \mathrm{Aut}_{Gp}(N)$ be the group homomorphism induced by the conjugation act of H on N, ie $\alpha(h)(n) = h n h^{-1} \in N \quad \forall h \in H \quad \forall n \in N$.

Next, we set $G = N \rtimes_\alpha H$ as in Lemma §23.1

Proposition:
$$\begin{aligned} f: G &\longrightarrow \mathcal{G} \\ (n,h) &\longmapsto nh \end{aligned}$$
is a group isomorphism.

Proof: First, we check this is a group homomorphism:

$$f((n_1,h_1) *_\alpha (n_2,h_2)) = f((n_1 \alpha(h_1)(n_2), h_1 h_2)) = n_1 \alpha(h_1)(n_2) \cdot h_1 h_2$$
$$= n_1 h_1 n_2 h_1^{-1} h_1 h_2 = n_1 h_1 n_2 h_2 = f((n_1,h_1))\, f((n_2,h_2))$$

- f is surjective because $NH = \mathcal{G}$.
- f is injective because $f(n,h) = nh = e \iff n = h^{-1} \in N \cap H = \{e\}$

so $n = e = h$. Thus $\mathrm{Ker}(f) = \{(e,e)\}$.

Summary:
$$\left\{ \begin{array}{c} \text{Semi-direct products} \\ \text{of } H \text{ \& } N \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Group homomorphisms} \\ H \longrightarrow \mathrm{Aut}_{Gp}(N) \end{array} \right\}$$

$$G = N \rtimes_\alpha H \longleftarrow\!\shortmid \quad \alpha$$

Next, we state a Corollary of the 3[rd] Isomorphism Thm when $G = H\cdot N$, $H \cap N = \{e\}$

<u>Corollary</u>: Let G be a semi-direct product of H & N. Then, the natural projection

$$\begin{aligned} \pi : G &\longrightarrow G/N \\ g &\longmapsto gN \end{aligned}$$

restricted to H induces an isomorphism
$$\begin{aligned} \pi|_H : H &\xrightarrow{\sim} G/N \\ h &\longmapsto hN \end{aligned}$$

<u>Proof</u>:
- $\pi|H$ is group homomorphism
- $G = H\cdot N$ so $\pi|_H$ is surjective
- $H \cap N$ so $\pi|_H$ is injective

□

<u>Q</u>: What does this mean?

<u>A</u>: For every coset $\bar{g} \in G/N$, we can find a representative $\bar{g} = \sigma_g N$ such that
- $\mathrm{ord}_G(\sigma_g) = \mathrm{ord}_{G/N}(\bar{g})$
- $\sigma_{g_1 g_2} = \sigma_{g_1}\sigma_{g_2}$

Namely: $\sigma_g = (\pi|_H)^{-1}(\bar{g}) \in H \subseteq G$.

<u>Summary II</u>: $N \trianglelefteq G$ and $H \leq G$

- 3[rd] Iso Theorem: $H \leq G \implies H/H\cap N \simeq H\cdot N/N$
- If $NH = G$, $H \cap N = \{e\}$ $\implies$

$$\begin{tikzcd} \pi: G \arrow[r, two heads] & G/N \arrow[l, bend left, dashed] \\ \sigma_g & \bar{g} \arrow[l, mapsto] \end{tikzcd}$$

"lifting"

<u>Next Time</u>: Compute $\mathrm{Aut}_{Gp}(N)$ for some groups N