Lecture XXVII: Classifying groups of small order

§27.1 Summary:

- $\mathrm{Aut}_{Gp}(\mathbb{Z}/n\mathbb{Z})$ is an abelian group with $\varphi(n)$ elements (Euler's φ function)
- $n = p_1^{a_1} p_2^{a_2} \cdots p_\ell^{a_\ell} \Rightarrow \mathbb{Z}/n\mathbb{Z} \simeq \mathbb{Z}/p_1^{a_1}\mathbb{Z} \times \cdots \times \mathbb{Z}/p_\ell^{a_\ell}\mathbb{Z}$
- $\varphi(n) = \varphi(p_1^{a_1}) \cdots \varphi(p_\ell^{a_\ell}) \Rightarrow \mathrm{Aut}_{Gp}(\mathbb{Z}/n\mathbb{Z}) \simeq \mathrm{Aut}_{Gp}(\mathbb{Z}/p_1^{a_1}\mathbb{Z}) \times \cdots \times \mathrm{Aut}_{Gp}(\mathbb{Z}/p_\ell^{a_\ell}\mathbb{Z})$
- $\varphi(p^r) = p^{r-1}(p-1)$
  - [p odd] $\mathrm{Aut}_{Gp}(\mathbb{Z}/p^r\mathbb{Z})$ is cyclic $(= \langle 1+p \rangle \leq (\mathbb{Z}/p^r\mathbb{Z})^*)$
  - [p even] . $\mathrm{Aut}_{Gp}(\mathbb{Z}/2^r\mathbb{Z}) \simeq \underset{\underset{\langle 2^{r-1} \rangle}{\parallel}}{\mathbb{Z}/2\mathbb{Z}} \times \underset{\underset{\langle 5 \rangle \text{ in } (\mathbb{Z}/2^r\mathbb{Z})^*}{\parallel}}{\mathbb{Z}/2^{r-2}\mathbb{Z}}$ if $r \geq 3$
    . $\mathrm{Aut}_{Gp}(\mathbb{Z}/2\mathbb{Z}) = \{e\}$, $\mathrm{Aut}_{Gp}(\mathbb{Z}/2^2\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z}$.

§27.2 Example: Classify all groups of order 18:

Let G be a group with 18 elements $(18 = 2 \cdot 3^2)$

By Sylow Theorems, we have $P \leq G$ a subgroup with 2 elements
$Q \leq G$ ——— 9 ———

Note: • $Q \trianglelefteq G$ because it has index 2.
• Alternative reason $n_3 = 1 \bmod 3$ & $n_3 \mid 2 \Rightarrow n_3 = 1$.

Conclusion: (1) $P \leq G$, $Q \trianglelefteq G$

(2) $P \cap Q = \{e\}$ because $|P|$ & $|Q|$ are coprime

(3) $PQ \underset{\substack{\downarrow \\ \text{because } Q \trianglelefteq G}}{=} QP = G$ $(|PQ|$ divisible by $2 = |P|$ & $9 = |Q|)$

$\Rightarrow G \cong Q \rtimes_\alpha P$ for some $\alpha : P \longrightarrow \mathrm{Aut}_{Gp}(Q)$ gp homomorphism.

We know $\alpha(h) = \mathrm{Conj}(h)$ $\forall h \in P$.

• **Options for P & Q:** $P \simeq \mathbb{Z}/2\mathbb{Z}$; $Q \cong \mathbb{Z}/9\mathbb{Z}$ or $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$
by Proposition §18.2.

We treat these 2 cases separately

CASE 1: $P \simeq \mathbb{Z}/2\mathbb{Z}$; $Q \simeq \mathbb{Z}/9\mathbb{Z}$ cyclic

$\mathrm{Aut}_{Gp}(Q) = \mathrm{Aut}_{Gp}(\mathbb{Z}/3^2\mathbb{Z})$ is cyclic of order $3 \cdot 2 = 6$. Pick σ a generator

$\Rightarrow \mathrm{Aut}_{Gp}(Q) = \{id, \sigma, \sigma^2, \sigma^3, \sigma^4, \sigma^5\}$ (Eg $\sigma: \mathbb{Z}/9\mathbb{Z} \longrightarrow \mathbb{Z}/9\mathbb{Z}$, $1 \longmapsto 5 \bmod 9$)

Group homomorphisms $\underset{\{0,1\}}{P} \xrightarrow{\alpha} \underset{\alpha(1)}{\mathrm{Aut}_{Gp}(Q)}$

- If $\alpha(1) = id \Rightarrow \alpha$ is trivial, and $G \simeq P \times Q \simeq \mathbb{Z}/18\mathbb{Z}$
- If $\alpha(1) \neq id$, then $\alpha(1)$ has order 2 in $\mathrm{Aut}_{Gp}(Q)$

The only possibility is $\alpha(1) = \sigma^3$ so $\sigma^3 : Q \longrightarrow Q$

$1 \bmod 9 \longmapsto 5^3 \equiv -1 \bmod 9$. (additive notation)

"inverse of 1."

Thus $Q \rtimes_\alpha P$ can be explicitly described as follows:

if $P = \langle x \rangle$, $Q = \langle y \rangle$ so that $x^2 = e_P$ (id in P) then,

$y^9 = e_Q$ (id in Q)

$$(e_Q, x) *_\alpha (y, e_P) = (\alpha(x)(y), x) = (y^{-1}, x) = (y^{-1}, e_P) *_\alpha (e_Q, x)$$

Hence, identifying (e_Q, x) with x and (y, e_P) with y, we get

$$G \simeq \langle x, y \mid x^2 = y^9 = 1, xy = y^{-1}x \rangle = \langle x, y \mid x^2 = y^9 = 1, xyx = y^{-1} \rangle \simeq D_{18}$$

Conclude: $G \simeq \mathbb{Z}/18\mathbb{Z}$. or $G \simeq D_{18}$

CASE 2: $P \simeq \mathbb{Z}/2\mathbb{Z}$; $Q \simeq \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$

Claim: $\mathrm{Aut}_{Gp}(Q)$ can be viewed as 2×2 invertible matrices with entries from $\mathbb{F}_3$ ($\simeq \mathbb{Z}/3\mathbb{Z}$)

Pf/ Pick $\sigma \in \mathrm{Aut}_{Gp}(Q)$. Then, $(a \pmod 3, b \pmod 3) \in Q$

$\downarrow \sigma$

$(\alpha a + \beta b \pmod 3, \gamma a + \delta b \pmod 3) \in Q$

where $(\alpha \pmod 3, \gamma \pmod 3) = \sigma(\bar{1}, 0)$

$(\beta \pmod 3, \delta \pmod 3) = \sigma(0, \bar{1})$

Reason: $\sigma(\bar{a}, \bar{b}) = \sigma((\bar{a}, 0) + (0, \bar{b})) = \sigma((\bar{a}, 0)) + \sigma((0, \bar{b}))$ $\forall a, b$

But $\sigma((\bar{a}, 0)) = \sigma(a(\bar{1}, 0)) = a\,\sigma(\bar{1}, 0)$

$\sigma((0, \bar{b})) = \sigma(b(0, \bar{1})) = b\,\sigma(0, \bar{1})$

$\Rightarrow \quad \sigma(\bar{a},\bar{b}) = a(\bar{\alpha},\bar{\gamma}) + b(\bar{\beta},\bar{\delta}) = (\overline{\alpha a + b\beta}, \overline{\gamma a + \delta b}) = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}$

So we identify σ with $\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$: 2×2 matrix with entries from $\mathbb{F}_3$.

Note: Composition of elements of $\mathrm{Aut}_{Gp}(Q)$ = matrix multiplication

$\sigma = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$ is an iso $\Longleftrightarrow$ $\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$ is invertible.

So $\mathrm{Aut}_{Gp}(Q) \simeq GL_2(\mathbb{F}_3)$ $\square$

Recall: $|GL_2(\mathbb{F}_3)| = (3^2-1)(3^2-3) = 8\cdot 6 = 48$.

Now, we are looking for $\alpha : P \longrightarrow \mathrm{Aut}_{Gp}(Q) = GL_2(\mathbb{F}_3)$

$1 \longmapsto X$

We have $\alpha(1)^2 = \alpha(2\cdot 1) = \alpha(0) = id_Q$, so $X^2 = I_2$.

In particular $\det(X)^2 = 1 \Rightarrow \det(X) = \pm 1$

$X = X^{-1}$

We have 2 options for $X = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$

- If $\det(X) = 1 \Rightarrow X = X^{-1}$ gives $\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} = \frac{1}{1}\begin{bmatrix} \delta & -\beta \\ -\gamma & \alpha \end{bmatrix} \Longleftrightarrow \begin{matrix} \alpha = \delta \\ \beta = -\beta \\ \gamma = -\gamma \end{matrix}$

  Thus $X = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix}$ with $\alpha^2 = 1$. $\alpha \in \mathbb{F}_3^*$

  We have 2 options $X = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$ or $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = -I_2$ (2)

- If $\det(X) = -1 \Rightarrow X = X^{-1}$ gives $\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} = \frac{1}{-1}\begin{bmatrix} \delta & -\beta \\ -\gamma & \alpha \end{bmatrix} \Longleftrightarrow \alpha = -\delta$

  Thus $X = \begin{bmatrix} \alpha & \beta \\ \gamma & -\alpha \end{bmatrix}$ with $-\alpha^2 - \beta\gamma = -1$.

  We have several options

  - $\alpha = \delta = 0 \Rightarrow -\beta\gamma = -1$ gives $X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ or $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ (2)
  - $\alpha = 1, \delta = -1 \Rightarrow -1-\beta\gamma = -1$ gives $X = \begin{bmatrix} 1 & x \\ 0 & -1 \end{bmatrix}$ or $\begin{bmatrix} 1 & 0 \\ x & -1 \end{bmatrix}$ $x \in \mathbb{F}_3$ (6)
  - $\alpha = -1, \delta = 1 \Rightarrow -1-\beta\gamma = -1$ gives $X = \begin{bmatrix} -1 & x \\ 0 & 1 \end{bmatrix}$ or $\begin{bmatrix} -1 & 0 \\ x & 1 \end{bmatrix}$ $x \in \mathbb{F}_3$ (6)

TOTAL = 16 options.

However, there are only 3 conjugacy classes (ie same linear transf. written in different bases)

Claim 1: All 14 cases corresponding to $\begin{array}{l}\alpha+\delta=0\\ \alpha\delta-\beta\gamma=-1\end{array}$ are conjugate to each other

Pf/ We show they are all conjugate to $\begin{bmatrix}1&0\\0&-1\end{bmatrix}$

First, we compute the stabilizer of $\begin{bmatrix}1&0\\0&-1\end{bmatrix}$ under conjugation.

$$\begin{bmatrix}1&0\\0&-1\end{bmatrix}=\begin{bmatrix}a&b\\c&d\end{bmatrix}\begin{bmatrix}1&0\\0&-1\end{bmatrix}\begin{bmatrix}a&b\\c&d\end{bmatrix}^{-1}\iff\begin{bmatrix}1&0\\0&-1\end{bmatrix}\begin{bmatrix}a&b\\c&d\end{bmatrix}=\begin{bmatrix}a&b\\c&d\end{bmatrix}\begin{bmatrix}1&0\\0&-1\end{bmatrix}$$

$$\begin{bmatrix}a&b\\-c&-d\end{bmatrix}=\begin{bmatrix}a&-b\\c&-d\end{bmatrix}\iff b=c=0.$$

$$W=\mathrm{Stab}_{GL_2(\mathbb{F}_3)}\left(\begin{bmatrix}1&0\\0&-1\end{bmatrix}\right)=\left\{\begin{bmatrix}a&0\\0&d\end{bmatrix}:a,d\neq0\right\}$$

Next, we compute representatives of $GL_2(\mathbb{F}_3)/W$. We have 3 cases to consider:

$$\begin{bmatrix}a&b\\c&d\end{bmatrix}W=gW$$

(A). If $c=0\Rightarrow\begin{bmatrix}a&b\\0&d\end{bmatrix}=\begin{bmatrix}1&b/d\\0&1\end{bmatrix}\begin{bmatrix}a&0\\0&d\end{bmatrix}$ because $d\neq0$ $\Rightarrow g=\begin{bmatrix}1&b/d\\0&1\end{bmatrix}$

(B). If $b=0\Rightarrow\begin{bmatrix}a&0\\c&d\end{bmatrix}=\begin{bmatrix}1&0\\c/a&1\end{bmatrix}\begin{bmatrix}a&0\\0&d\end{bmatrix}$ —— $a\neq0$ $\Rightarrow g=\begin{bmatrix}1&0\\c/a&1\end{bmatrix}$

(C). If $c,b\neq0\Rightarrow\begin{bmatrix}a&b\\c&d\end{bmatrix}=\begin{bmatrix}a/c&1\\1&d/b\end{bmatrix}\begin{bmatrix}c&0\\0&b\end{bmatrix}$ $\Rightarrow g=\begin{bmatrix}a/c&1\\1&d/b\end{bmatrix}$

$\Rightarrow\begin{bmatrix}a&b\\c&d\end{bmatrix}\begin{bmatrix}1&0\\0&-1\end{bmatrix}\begin{bmatrix}a&b\\c&d\end{bmatrix}^{-1}=g\begin{bmatrix}1&0\\0&-1\end{bmatrix}g^{-1}$ where g is as above.

(A) $$\begin{bmatrix}1&x\\0&1\end{bmatrix}\begin{bmatrix}1&0\\0&-1\end{bmatrix}\begin{bmatrix}1&x\\0&1\end{bmatrix}^{-1}=\begin{bmatrix}1&x\\0&1\end{bmatrix}\begin{bmatrix}1&0\\0&-1\end{bmatrix}\begin{bmatrix}1&-x\\0&1\end{bmatrix}=\begin{bmatrix}1&-x\\0&-1\end{bmatrix}\begin{bmatrix}1&-x\\0&1\end{bmatrix}=\begin{bmatrix}1&-2x\\0&-1\end{bmatrix}$$

$=\begin{bmatrix}1&x\\0&-1\end{bmatrix}$ for $x\in\mathbb{F}_3$. $\Rightarrow$ we get half of the matrices in row 2.

(B) $$\begin{bmatrix}1&0\\x&1\end{bmatrix}\begin{bmatrix}1&0\\0&-1\end{bmatrix}\begin{bmatrix}1&0\\x&1\end{bmatrix}^{-1}=\begin{bmatrix}1&0\\x&1\end{bmatrix}\begin{bmatrix}1&0\\0&-1\end{bmatrix}\begin{bmatrix}1&0\\-x&1\end{bmatrix}=\begin{bmatrix}1&0\\x&-1\end{bmatrix}\begin{bmatrix}1&0\\-x&1\end{bmatrix}=\begin{bmatrix}1&0\\2x&-1\end{bmatrix}$$

$=\begin{bmatrix}1&0\\-x&-1\end{bmatrix}$ for $x\in\mathbb{F}_3$ $\Rightarrow$ we get the other half of the matrices in row 2.

(C) $$\begin{bmatrix}x&1\\1&y\end{bmatrix}\begin{bmatrix}1&0\\0&-1\end{bmatrix}\begin{bmatrix}x&1\\1&y\end{bmatrix}^{-1}=\begin{bmatrix}x&1\\1&y\end{bmatrix}\begin{bmatrix}1&0\\0&-1\end{bmatrix}\frac{1}{\Delta}\begin{bmatrix}y&-1\\-1&x\end{bmatrix}=\begin{bmatrix}x&-1\\1&-y\end{bmatrix}\frac{1}{\Delta}\begin{bmatrix}y&-1\\-1&x\end{bmatrix}$$

$\Delta=xy-1\neq0$

$$=\frac{1}{\Delta}\begin{bmatrix}xy+1&-2x\\2y&-(xy+1)\end{bmatrix}=\frac{1}{\Delta}\begin{bmatrix}\Delta+2&x\\-y&-(\Delta+2)\end{bmatrix}=\frac{1}{\Delta}\begin{bmatrix}\Delta-1&x\\-y&-\Delta+1\end{bmatrix}=M$$

- If $x=0\Rightarrow\Delta=-1$ & $M=\frac{1}{-1}\begin{bmatrix}-2&0\\-y&2\end{bmatrix}=\begin{bmatrix}-1&0\\y&1\end{bmatrix}$ for $y\in\mathbb{F}_3$

We get the second half of the matrices in row 3.

- If $y=0 \Rightarrow \Delta=-1$ & $M=\frac{1}{-1}\begin{bmatrix}-2 & -2x\\ 0 & 2\end{bmatrix}=\begin{bmatrix}-1 & -x\\ 0 & 1\end{bmatrix}$ for $x\in\mathbb{F}_3$.

  We get the first half of the matrices in row 3.

- If $x=1 \Rightarrow y=-1$ because $\Delta\neq 0$. Indeed $\Delta=-1-1=-2=1$

  Then $M=\frac{1}{1}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$ We get the first matrix in row 1.

- If $x=-1 \Rightarrow y=1$ because $\Delta\neq 0$. Indeed, $\Delta=-1-1=-2=1$

  Then $M=\frac{1}{1}\begin{bmatrix}0 & -1\\ -1 & 0\end{bmatrix}=\begin{bmatrix}0 & -1\\ -1 & 0\end{bmatrix}$ We get the remaining matrix in row 1. $\square$

Conclusion: Our options for $\alpha: P\longrightarrow \mathrm{Aut}_{Gp}(Q)=GL_2(\mathbb{F}_3)$ are 3 (up to conjugation), namely $\sigma(\text{gen of } P)=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ or $\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}$ or $\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$

This gives 3 more groups. We use multiplicative notation since they need not be abelian.

Recall: $\alpha(h)(n)=hnh^{-1}$ in $Q\rtimes_\alpha P$ $h\in P$, $n\in Q$.

(1) $\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ $\rightsquigarrow$ $G\simeq \underbrace{(\mathbb{Z}/3\mathbb{Z})^2}_{\langle y_1\rangle\times\langle y_2\rangle}\times\underbrace{\mathbb{Z}/2\mathbb{Z}}_{\langle x\rangle}\simeq\left\langle y_1,y_2,x \,\middle|\, \begin{array}{l} x^2=y_1^3=y_2^3=e\\ y_1y_2=y_2y_1\\ \boxed{\begin{array}{l}xy_1x=y_1\\ xy_2x=y_2\end{array}}\end{array}\right\rangle$

(direct product)

(2) $\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}$ $\rightsquigarrow$ $G=(\mathbb{Z}/3\mathbb{Z})^2\rtimes_\alpha \mathbb{Z}/2\mathbb{Z}\simeq\left\langle y_1,y_2,x \,\middle|\, \begin{array}{l} x^2=y_1^3=y_2^3=e\\ y_1y_2=y_2y_1\\ \boxed{\begin{array}{l}xy_1x=y_1^{-1}\\ xy_2x=y_2^{-1}\end{array}}\end{array}\right\rangle$

For $y\in Q$, we get:

$(*)\ (e,x)*_\alpha(y,e)*_\alpha(e,x)=(\alpha(x)(y),x)*_\alpha(e,x)=(\alpha(x)(y)\underbrace{\alpha(x)(e)}_{e},\underset{\overset{\|}{e}}{x^2})$

- If $y=y_1$, $\alpha(x)(y_1)=\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix}=\begin{bmatrix}-1\\ 0\end{bmatrix}=y_1^{-1}$. $\Rightarrow xy_1x=y_1^{-1}$

  additive notation — multiplicative notation

- If $y=y_2$, $\alpha(x)(y_2)=\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}0\\ 1\end{bmatrix}=\begin{bmatrix}0\\ -1\end{bmatrix}=y_2^{-1}$ $\Rightarrow xy_2x=y_2^{-1}$

③ $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \rightsquigarrow G = (\mathbb{Z}/3\mathbb{Z})^2 \rtimes_\alpha \mathbb{Z}/2\mathbb{Z} \cong \left\langle y_1, y_2, x \;\middle|\; \begin{array}{l} x^2 = y_1^3 = y_2^3 = e \\ y_1 y_2 = y_2 y_1 \\ \boxed{\begin{array}{l} x y_1 x = y_1 \\ x y_2 x = y_2^{-1} \end{array}} \end{array} \right\rangle$

Using (*) we get $\alpha(x)(y_1) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = y_1 \Rightarrow x y_1 x = y_1$.

$\alpha(x)(y_2) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} = y_2^{-1} \Rightarrow x y_2 x = y_2^{-1}$

Q: Why can we ignore conjugate matrices?

A: There is a general Lemma that gives isomorphic $N \rtimes_\alpha H$.

Lemma: Let H, N be 2 groups & $\alpha, \beta : H \longrightarrow \mathrm{Aut}_{Gp}(N)$ two group homomorphism.

Assume $\exists\, T \in \mathrm{Aut}_{Gp}(N)$ s.t. $\alpha(h)(n) = T(\beta(h)(T^{-1}(n)))$ $\forall h \in H, n \in N$

Then, $N \rtimes_\alpha H \simeq N \rtimes_\beta H$.

Proof: Define $f: \begin{array}{ccc} N \rtimes_\beta H & \longrightarrow & N \rtimes_\alpha H \\ (n, h) & \longrightarrow & (T(n), h) \end{array}$

(1) f is a group homomorphism: Pick $n_1, n_2 \in N$, $h_1, h_2 \in H$.

$f((n_1, h_1) *_\beta (n_2, h_2)) = f(n_1 \beta(h_1)(n_2), h_1 h_2)) = (T(n_1 \beta(h_1)(n_2)), h_1 h_2)$

$\underset{\substack{\uparrow \\ T \text{ gp hom}}}{=} (T(n_1)\, T(\beta(h_1)(n_2)), h_1 h_2)$

$f(n_1, h_1) *_\alpha f(n_2, h_2) = (T(n_1), h_1) *_\alpha (T(n_2), h_2) = (T(n_1)\, \alpha(h_1)(T(n_2)), h_1 h_2)$

$\underset{\substack{\downarrow \\ \text{relation between } \alpha, T, \beta}}{=} (T(n_1)\, T(\beta(h_1)(T^{-1}(T(n_2))), h_1 h_2) = (T(n_1)\, T(\beta(h_1)(n_2)), h_1 h_2)$. ✓

(2) f is injective $(T(n), h) = (e_N, e_H) \underset{\substack{\downarrow \\ T \text{ iso}}}{\Leftrightarrow} n = e_N$ & $h = e_H$.

Thus $\mathrm{Ker}(f) = \{(e_N, e_H)\}$

(3) f is surjective $f(T^{-1}(n), h) = (n, h)$.

Conclusion: If matrix of $\alpha(h)$ is conjugate to $\beta(h)$, the semidirect products are isomorphic!