Lecture XXVIII: Composition Series

Recall the "plan to study finite groups":

(1) Classification of finite simple groups (simple gp = a gp with no proper non-trivial normal subgroups)

(2) General result on how to fit groups together.

TODAY: We focus on (2)

§28.1 Composition Series:

Definition: Let G be a group. A composition series of G is a finite sequence of normal subgroups

$$\Sigma: \quad G = G_0 \trianglerighteq G_1 \trianglerighteq G_2 \cdots \trianglerighteq G_n = \{e\} \qquad (G_{i+1} \trianglelefteq G_i \ \forall i=0,\dots,n-1)$$

Name: The sequence of successive quotients $\{G_i/G_{i+1}\}_{i=0}^{n-1}$ are often called the graded pieces of Σ (of G via Σ)

Sometimes we will use the notation $gr_i^{\Sigma}(G)$ to label the "i^{th} graded piece of G G_i/G_{i+1} obtained via the composition series Σ"

Definition: We say the composition series Σ is strict if $G_i \trianglerighteq\!\!\!\!\!/\; G_{i+1}$, i.e. $G_i \ntrianglerighteq G_{i+1}$ wait

That is, $gr_i^{\Sigma}(G) \neq \{e\}$ $\forall i$

Example 1: $G = D_{2n}$ dihedral group

$\Sigma_1: \ D_{2n} = G_0 \trianglerighteq G_1 = \langle r \rangle \simeq \mathbb{Z}/n\mathbb{Z} \trianglerighteq \{e\}$ is a strict composition series

Graded pieces: $\{\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}\}$

Note: We can refine Σ by adding more terms between G_1 & $\{e\}$. This will correspond to a composition series of $\mathbb{Z}/n\mathbb{Z}$

Example 2: $\Sigma: S_n \trianglerighteq A_n \trianglerighteq \{e\}$ is a strict composition series for S_n.

Graded pieces $\{\mathbb{Z}/2\mathbb{Z}, A_n\}$

$A_n = \langle \sigma : \sigma \in S_n \text{ is a 3-cycle} \rangle$ (alternating group) $|A_n| = \frac{n!}{2}$ (future lecture)

We will see that Σ cannot be refined in a strict way if $n \geq 5$. Reason: A_n is simple if $n \geq 5$.

§28.2 Refinements of composition series:

Let G be a group and let there be two composition series

$$\Sigma_1 : G = G_0 \trianglerighteq G_1 \trianglerighteq \cdots \trianglerighteq G_n = \{e\}$$

$$\Sigma_2 : G = G'_0 \trianglerighteq G'_1 \trianglerighteq \cdots \trianglerighteq G'_m = \{e\}$$

Definition: We say Σ_2 is finer than Σ_1 (or Σ_2 is a refinement of Σ_1) if Σ_1 is obtained from Σ_2 by omitting a few terms.

Equivalently Σ_1 is a coarsening of Σ_2.

Example: $\Sigma_1 : G_0 = D_{12} \trianglerighteq G_1 = \langle r \rangle \trianglerighteq \{e\}$, with $\langle r \rangle \cong_{12} \mathbb{Z}/6\mathbb{Z}$

$$\Sigma_2 : G'_0 = D_{12} \trianglerighteq G'_1 = \langle r \rangle \trianglerighteq G'_2 = \langle r^2 \rangle \trianglerighteq \{e\}$$

with $\langle r \rangle \cong_{12} \mathbb{Z}/6\mathbb{Z}$, $\langle r^2 \rangle \cong_{12} \mathbb{Z}/3\mathbb{Z}$

Σ_2 is finer than Σ_1 (or Σ_1 is a "coarsening of Σ_2", ie some terms dropped)

$A \trianglelefteq B \trianglelefteq C$ does NOT imply $A \trianglelefteq C$.

Example: Take any non-trivial group H and form a semidirect product.

$$C = G = (H \times H) \rtimes_\alpha \mathbb{Z}/2\mathbb{Z} \qquad (\alpha(s)((x,y)) = (y,x))$$

$$\mathbb{Z}/2\mathbb{Z} = \langle s \rangle$$

$$\begin{aligned} C &= G \\ &\ \ \text{I}\triangledown \\ B &= H \times H \\ &\ \ \text{I}\triangledown \\ A &= H \times \{e\} \simeq H \end{aligned}$$

$A \ntrianglelefteq C$ since $s *_\alpha h *_\alpha s = ((e,e),s) *_\alpha ((h,e),e) *_\alpha ((e,e),s)^{-1}$

$\left(\begin{array}{l} s = ((e,e),s) \in C \\ h = ((h,e),e) \in A \end{array} \right)$

$$= ((e,e)\,\alpha(s)(h,e), s) *_\alpha (\alpha(s^{-1})((e,e)^{-1}), s^{-1})$$

$$= ((e,h),s) *_\alpha ((e,e), s^{-1}) = ((e,h)\,\alpha(s)(e,e), ss^{-1}) = ((e,h),e) \notin A.$$

Consequence: Dropping terms of a composition series need not give a composition series.

§28.3 Equivalence of composition series:

Let G and H be two groups an let

$$\Sigma : G = G_0 \trianglerighteq G_1 \trianglerighteq \cdots \trianglerighteq G_n = \{e\}$$

$$\Sigma' : H = H_0 \trianglerighteq H_1 \trianglerighteq \cdots \trianglerighteq H_m = \{e\}$$

be two composition series.

Definition: We say Σ and Σ' are *equivalent* if $m = n$ and we have a permutation $\sigma \in S_n$ such that $G_i/G_{i+1} \simeq H_{\sigma(i)}/H_{\sigma(i)+1}$ (S_n = permutations of $\{0, 1, \dots, n-1\}$)

(ie the sets $\{G_i/G_{i+1}\}_{i=0}^{n-1}$ and $\{H_j/H_{j+1}\}_{j=0}^{n-1}$ are "the same")

Remark: We didn't require G and H be isomorphic. In fact, they need not be, as the next example shows.

Example: Consider $G = D_8$ with the following composition series:

$$\Sigma: D_8 = G = G_0 \trianglerighteq G_1 = \langle r \rangle \simeq \mathbb{Z}/4\mathbb{Z} \trianglerighteq G_2 = \langle r^2 \rangle \simeq \mathbb{Z}/2\mathbb{Z} \trianglerighteq G_3 = \{e\}$$

Take $H = Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ With operations $(-1)^2 = 1$

$$(\pm i)^2 = (\pm j)^2 = (\pm k)^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j$$

$$\Sigma': Q_8 = H = H_0 \trianglerighteq H_1 = \{\pm i, \pm 1\} = \underset{\substack{\wr\wr \\ \mathbb{Z}/4\mathbb{Z}}}{\langle i \rangle} \trianglerighteq H_2 = \{\pm 1\} = \underset{\substack{\wr\wr \\ \mathbb{Z}/2\mathbb{Z}}}{\langle -1 \rangle} \trianglerighteq H_3 = \{e\}$$

We get the same graded pieces ($\mathbb{Z}/2\mathbb{Z}$ 3 times), so Σ and Σ' are equivalent. However $D_8 \not\simeq Q_8$ because D_8 has 2 elements of order 4 (r & r^3) but Q_8 has 6 elements of order 4 ($\pm i, \pm j, \pm k$).

§28.4 Jordan-Hölder series:

Definition: A composition series Σ of a group G is said to be *Jordan-Hölder* if it is strict and any strict composition series Σ' finer that Σ is equal to Σ

In short: a Jordan-Hölder series is maximal among all strict composition series

Note: If $|G| = \infty$ it may not admit any Jordan-Hölder series

Example: $\mathbb{Z} = G = G_0$. Let $\Sigma: \mathbb{Z} \ntrianglerighteq Z_1 \ntrianglerighteq Z_2 \ntrianglerighteq \cdots \ntrianglerighteq Z_{m-1} \ntrianglerighteq Z_m = \{e\}$

Since $\mathbb{Z}$ is cyclic, each Z_i is cyclic, so $Z_{m-1} = \langle r \rangle, r > 1$. But we can refine Σ via $Z_{m-1} \ntrianglerighteq \langle 2r \rangle \ntrianglerighteq Z_m \Rightarrow \Sigma$ is not JH.