Lecture XXIX: Jordan-Hölder series

Last time we introduced composition series and Jordan-Hölder series:

- A composition series of a group G is a descending sequence of normal subgroups:

$$G = G_0 \trianglerighteq G_1 \trianglerighteq G_2 \trianglerighteq \cdots \trianglerighteq G_n = \{e\}$$

(Remember: G_{j+2} need not be normal in G_j $\forall j$)

- Graded pieces associated to Σ: $gr_i^{\Sigma}(G) = G_i/G_{i+1}$ $(i=0,1,\dots,n-1)$
- Length of $\Sigma = n$.
- Σ' is a refinement of Σ if Σ' is a composition series obtained from Σ by adding normal subgroups (equivalent Σ is obtained from Σ' by omitting terms). In particular, we have $\text{length}(\Sigma) \leq \text{length}(\Sigma')$
- Σ_1 is equivalent to Σ_2 if (1) $\text{length}(\Sigma_1) = \text{length}(\Sigma_2)$
  (2) they have the same graded pieces (possibly up to permutation)

Example: $\mathbb{Z}/4\mathbb{Z} \trianglerighteq \mathbb{Z}/2\mathbb{Z} \trianglerighteq \{0\}$
$(\mathbb{Z}/2\mathbb{Z})^2 \trianglerighteq \mathbb{Z}/2\mathbb{Z} \times \{0\} \trianglerighteq \{(0,0)\}$
are equivalent, even though groups are NOT isomorphic.

- Σ is strict if $gr_i^{\Sigma}(G) \neq \{e\}$ $\forall i = 0, \dots, n-1$. Equivalently, $G_i \trianglerighteq\!\!\!\!\!\!/\;\; G_{i+1}$ $\forall i$.

- A Jordan-Hölder series is a strict composition series that is maximal relative to refinement among all strict composition series. Meaning: Σ is J-H if strict and for every Σ' strict and finer than Σ we have $\Sigma' = \Sigma$.

§29.1 Characterization & existence of J-H series:

Lemma: Let G be a group. A strict composition series $\Sigma: G = G_0 \gneq \cdots \gneq G_n = \{e\}$ is Jordan-Hölder if, and only if, each G_i/G_{i+1} is simple $\forall i = 0, \dots, n-1$

Proof: This follows from the 2^{nd} Isomorphism Theorem

($\Rightarrow$) We prove the contrapositive. If G_i/G_{i+1} is not simple for some i. Then $\exists \overline{N}$ non-trivial proper normal subgroup of G_i/G_{i+1}. By the Second Isomorphism Theorem applied

To $G_i \twoheadrightarrow G_i/G_{i+1}$ we have $N := \pi^{-1}(\bar{N}) \ntrianglelefteq G_i$ & $G_{i+1} \subsetneq N$. Since $G_{i+1} \trianglelefteq G_i$, then we have $G_{i+1} \ntrianglelefteq N$. So Σ is not maximal: we can refine it by replacing $G_i \ntrianglerighteq G_{i+1}$ with $G_i \ntrianglerighteq N \ntrianglerighteq G_{i+1}$. Thus, Σ is not Jordan-Hölder.

($\Leftarrow$) Again, we prove the contrapositive. If Σ is not Jordan-Hölder, we can refine it by some Σ'. In particular, for some i, we replace $G_i \ntrianglerighteq G_{i+1}$ by

$G_i \ntrianglerighteq G_i' \ntrianglerighteq \dots \ntrianglerighteq G_i^j = G_{i+1}$. with $j \geq 2$

Applying the canonical projection $\pi : G_i \longrightarrow G_i/G_{i+1}$, we obtain a chain

$$G_i/G_{i+1} \geq G_i'/G_{i+1} \geq \dots \geq G^j_i/G_{i+1} = \{e\}.$$

. Since $\ker \pi = G_{i+1}$, these inclusions are strict.

. Furthermore, since π is surjective, $G_i^j \trianglerighteq G_i^{j+1} \Rightarrow G_i^j/G_{i+1} \trianglerighteq G_i^{j+1}/G_{i+1}$.

In particular, $\{e\} \neq G_i'/G_{i+1} \ntrianglelefteq G_i/G_{i+1}$, so G_i/G_{i+1} is not simple. □

<u>Proposition</u>: Every finite G admits (at least) one Jordan-Hölder series.

<u>Proof:</u> We proceed by complete induction on $|G|$.

<u>Base Case</u>: $|G| = 1 \Rightarrow G = \{e\}$ is the Jordan Hölder series.

<u>Inductive Step</u>: Fix a finite group G & assume that the statement is true for all group H with $|H| < |G|$. We will build a composition series for G.

Choose a maximal normal proper subgroup of G, say N. (For example, $N \ntrianglelefteq G$ with $|N|$ maximal will satisfy this property.)

. If $N = \{e\}$, then G is simple or $G = \{e\}$, & hence the only composition series of G is $G \trianglerighteq \{e\}$

It is strict if $|G| > 1$ and maximal, so it is a Jordan-Hölder series if $|G| > 1$.

If $G = \{e\}$, $\Sigma: G = G_0$ is JH.

. Otherwise, $G = G_0 \ntrianglerighteq N \ntrianglerighteq \{e\}$ is a strict composition series and G/N is simple because $\left\{ \begin{array}{c} \text{Normal subgroups of} \\ G/N \end{array} \right\} \xleftrightarrow{\text{1-to-1}} \left\{ \begin{array}{c} \text{normal subgroups of } G \\ \text{containing } N \end{array} \right\} = \{G, N\}$ because N is max'l

by the 2^{nd} Isomorphism Theorem.

Now, $|N| < |G|$ so by (IH) $\exists\ N = N_0 \ntrianglerighteq N_1 \ntrianglerighteq \dots \ntrianglerighteq N_m = \{e\}$ Jordan-Hölder series. By the Lemma, we have N_i/N_{i+1} is simple $\forall i = 0, \ldots, n-1$.

Then, $\Sigma: G = G_0 \underset{\neq}{\trianglerighteq} G_1 = N \underset{\neq}{\trianglerighteq} G_2 = N_1 \underset{\neq}{\trianglerighteq} \cdots \underset{\neq}{\trianglerighteq} G_{m-1} = N_m = \{e\}$ is a strict composition series for G. Since, by construction, G_i/G_{i+1} is simple $\forall i$, the same Lemma ensures that Σ is Jordan-Hölder. $\square$

§29.2 Uniqueness of J-H series:

Our first result discussed how to build equivalent refinements of 2 composition series:

Theorem (Schreier): Let Σ_1 and Σ_2 be two composition series of a group G. Then, there exist refinements Σ_1' & Σ_2' of Σ_1 & Σ_2, respectively, such that Σ_1' is equivalent to Σ_2'.

Corollary: Any 2 J-H series of a given group G are equivalent.

Proof: Let Σ_1, Σ_2 J-H series of G. The Theorem produces refinements Σ_1' of Σ_1, Σ_2' of Σ_2 with Σ_1' is equivalent to Σ_2'.

But Σ_1 & Σ_2 are maximal among strict composition series of G. This means that for $i=1,2$ Σ_i' is either equal to Σ_i or Σ_i' is NOT strict.

Thus, if $\Sigma_1 = G = G_0 \underset{\neq}{\trianglerighteq} G_1 \underset{\neq}{\trianglerighteq} G_2 \underset{\neq}{\trianglerighteq} \cdots \underset{\neq}{\trianglerighteq} G_n = \{e\}$ then, necessarily, Σ_1' has the form

$$\Sigma_1': \quad G: \underbrace{G_0 = G_0 = \cdots G_0}_{k_0 \text{ times}} \underset{\neq}{\trianglerighteq} \underbrace{G_1 = G_1 = \cdots G_1}_{k_1 \text{ times}} \underset{\neq}{\trianglerighteq} G_2 \cdots \quad \text{and so on} \ldots$$

The same conclusion applies to Σ_2'.

In particular, after removing repetitions, Σ_1' & Σ_2' have the same non-trivial graded pieces because Σ_1' & Σ_2' are equivalent. These are precisely the same graded pieces as Σ_1 & Σ_2.

So $\{gr_j^{\Sigma_i}(G)\} = \{gr_j^{\Sigma_i'}(G) : 0 \leq j \leq \text{length}(\Sigma_i') - 1 \text{ with } gr_j^{\Sigma_i'}(G) \neq \{e\}\}$ for $i = 1, 2$.

In particular, $\text{length}(\Sigma_1) = \text{length}(\Sigma_2)$ & $\{gr_j^{\Sigma_1}(G)\} = \{gr_j^{\Sigma_2}(G)\}$ $\square$

Example: $\Sigma_1: \ \mathbb{Z}/6\mathbb{Z} \underset{\neq}{\trianglerighteq} \mathbb{Z}/3\mathbb{Z} \underset{\neq}{\trianglerighteq} \{0\}$

$\Sigma_2: \ \mathbb{Z}/6\mathbb{Z} \underset{\neq}{\trianglerighteq} \mathbb{Z}/2\mathbb{Z} \underset{\neq}{\trianglerighteq} \{0\}$

both are J-H series

		0th	1st
Associated graded pieces with respect to	Σ_1:	$\{\mathbb{Z}/2\mathbb{Z}$,	$\mathbb{Z}/3\mathbb{Z}\}$
	Σ_2:	$\{\mathbb{Z}/3\mathbb{Z}$,	$\mathbb{Z}/2\mathbb{Z}\}$

Proof of Theorem: Assume we are given 2 composition series Σ_1, Σ_2 of a group G. Write

Σ_1 : $G = H_0 \trianglerighteq H_1 \trianglerighteq \cdots \trianglerighteq H_n = \{e\}$

Σ_2 : $G = K_0 \trianglerighteq K_1 \trianglerighteq \cdots \trianglerighteq K_m = \{e\}$

Idea: Use Σ_2 to refine each $H_i \trianglerighteq H_{i+1}$ for each $i = 0, \dots, n-1$.

$$H_i \trianglerighteq H_{i+1} \quad \rightsquigarrow \quad \left.\begin{array}{c} H_i = (H_i \cap K_0)\, H_{i+1} \\ \trianglerighteq \\ (H_i \cap K_1)\, H_{i+1} \\ \trianglerighteq \\ \vdots \\ H_{i+1} = (H_i \cap K_m)\, H_{i+1} \end{array}\right\} \text{TOTAL of } (m+1) \text{ terms}$$

Claim 1: $(H_i \cap K_j)\, H_{i+1} \leq H_i \quad \forall j$

Pf/ $H_i \cap K_j \leq H_i$ & $H_{i+1} \trianglelefteq H_i \implies (H_i \cap K_j)\, H_{i+1} \leq H_i$ by Lemma §22.1

Claim 2: $(H_i \cap K_j)\, H_{i+1} \trianglerighteq (H_i \cap K_{j+1})\, H_{i+1} \quad \forall j$ $\cap K_j) H_{i+1} \leq H_i$ by Lemma §22.1

(we prove this on page 6)

We set $\boxed{\mathcal{L}_{i,j} := (H_i \cap K_j)\, H_{i+1}}$ for $\begin{array}{l} 0 \leq i \leq n-1 \\ 0 \leq j \leq m \end{array}$

- By Claim 2 : $\mathcal{L}_{i,j} \trianglerighteq \mathcal{L}_{i,j+1} \quad \forall i = 0, \dots, n-1 \quad \forall j = 0, \dots, m-1$

- By construction : $\mathcal{L}_{i,m} = \mathcal{L}_{i+1,0} = H_{i+1} \quad \forall i = 0, \dots, n-1$

We get

$$\begin{aligned} G = \mathcal{L}_{0,0} = \underline{\underline{H_0}} = K_0 &\trianglerighteq \mathcal{L}_{0,1} \trianglerighteq \mathcal{L}_{0,2} \trianglerighteq \cdots \trianglerighteq \mathcal{L}_{0,m} = \underline{\underline{H_1}} \\ H_1 = \mathcal{L}_{1,0} &\trianglerighteq \mathcal{L}_{1,1} \trianglerighteq \cdots \cdots \trianglerighteq \mathcal{L}_{1,m} = \underline{\underline{H_2}} \\ H_2 = \mathcal{L}_{2,0} &\trianglerighteq \cdots \\ &\vdots \\ H_n = \mathcal{L}_{n-1,0} &\trianglerighteq \cdots \cdots \trianglerighteq \mathcal{L}_{n-1,m} = \underline{\underline{H_n}} = \{e\} \end{aligned}$$

We let $\boxed{\Sigma'_1}$ be the above composition series after removing $\mathcal{L}_{1,0}, \mathcal{L}_{2,0}, \dots, \mathcal{L}_{n-1}$.

In particular Σ'_1 refines Σ_1 (all $\underline{\underline{H_i\text{'s}}}$ are part of Σ'_1) & $\text{length}(\Sigma'_1) = n \cdot (m+1) - n = nm$

Similarly, we can use Σ_1 to refine Σ_2. Set $\boxed{\mathcal{R}_{ij} = (K_j \cap H_i)\, K_{j+1}}$ $\begin{array}{l} 0 \leq j \leq m-1 \\ 0 \leq i \leq n \end{array}$

, Since $K_{j+1} \triangleleft K_j$ & $K_j \cap H_i \leq K_j$, we have $\mathcal{R}_{ij} \leq K_j$ $\forall i$ $\forall j$

- Analogously to claim 2 we have $\mathcal{R}_{i+1,j} \triangleleft \mathcal{R}_{ij}$
- $\mathcal{R}_{n,j} = K_{j+1} = \mathcal{R}_{0,j+1}$ $\forall j = 0, \dots, m-1$.

We get

$$G = K_{0,0} = \underline{\underline{K_0}} = H_0 \trianglerighteq \mathcal{R}_{1,0} \trianglerighteq \mathcal{R}_{2,0} \trianglerighteq \cdots \trianglerighteq \mathcal{R}_{n,0} \underline{\underline{=}} K_1$$

$$K_1 = \mathcal{R}_{0,1} \trianglerighteq \mathcal{R}_{2,1} \trianglerighteq \cdots \trianglerighteq \mathcal{R}_{n,1} \underline{\underline{=}} K_2$$

$$K_2 = \mathcal{R}_{0,2} \trianglerighteq \cdots$$

$$\vdots$$

$$K_{m,0} = \mathcal{R}_{0,m-1} \trianglerighteq \cdots \trianglerighteq \mathcal{R}_{n,m-1} = \underline{\underline{K_m}}$$

We let $\boxed{\Sigma_2'}$ be the above composition series after removing $\mathcal{R}_{0,1}, \mathcal{R}_{0,2}, \dots, \mathcal{R}_{0,m-1}$

In particular Σ_2' refines Σ_2 (all K_i's are part of Σ_2') & $\text{length}(\Sigma_2') = m(n+1) - m = nm$

Thus, to prove Σ_1' & Σ_2' are equivalent, we need to identify their associated graded pieces:

<u>Claim 3:</u> $\mathcal{L}_{i,j} \trianglerighteq \mathcal{L}_{i,j+1}$ (in Σ_1') and $\mathcal{R}_{i,j} \trianglerighteq \mathcal{R}_{i+1,j}$ (in Σ_2') give isomorphic quotients

Both Claim 2 & 3 are consequences of Zassenhaus' Lemma (below) □

In the proof there are 4 groups

$$\mathcal{L}_{i,j} = (H_i \cap K_j) H_{i+1} \qquad \mathcal{R}_{i,j} = (H_i \cap K_j) K_{j+1}$$

$$\mathcal{L}_{i,j+1} = (H_i \cap K_{j+1}) H_{i+1} \qquad \mathcal{R}_{i+1,j} = (H_{i+1} \cap K_j) K_{j+1}$$

where $H_{i+1} \trianglelefteq H_i$ & $K_{j+1} \trianglelefteq K_j$

For notational convenience, we write

$$H := H_i \trianglerighteq H_{i+1} =: H'$$

$$K := K_j \trianglerighteq K_{j+1} =: K'$$

Zassenhaus Lemma: Let H, H', K and K' be four subgroups of G with $H' \trianglelefteq H$, $K' \trianglelefteq K$.

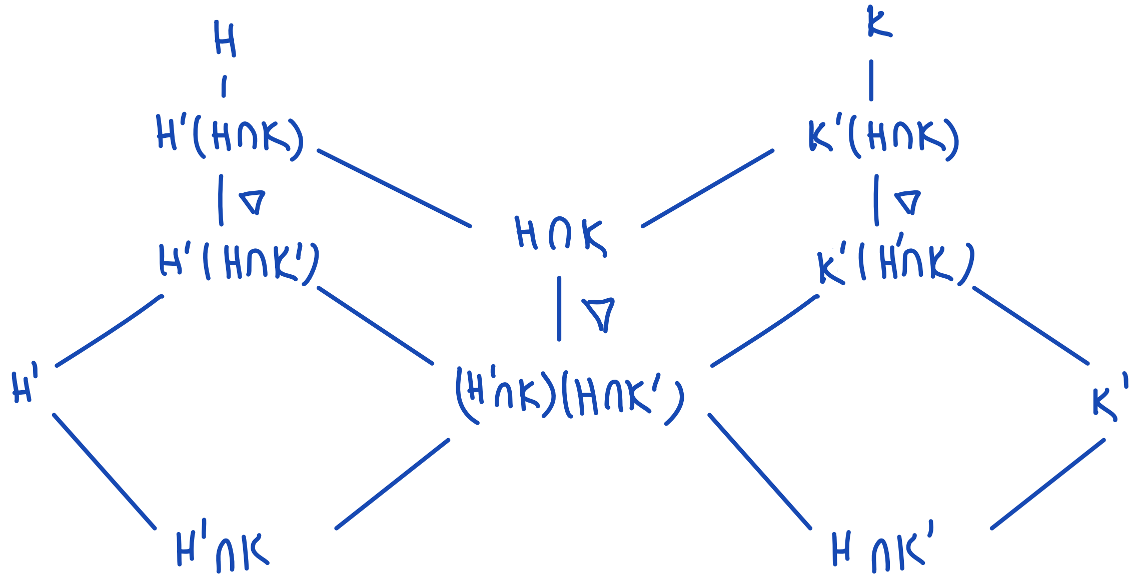
Then: (1) $(H\cap K')H' \trianglelefteq (H\cap K)\cdot H'$

$(K\cap H')K' \trianglelefteq (K\cap H)K'$

$(H\cap K')\cdot(H'\cap K) \trianglelefteq H\cap K$

(2) $$\frac{(H\cap K)H'}{(H\cap K')H'} \simeq \frac{(K\cap H)K'}{(K\cap H')K'} \simeq \frac{H\cap K}{(H\cap K')(H'\cap K)}$$

Remark: This statement is also known as the butterfly lemma.



Proof: (1) Since $H' \trianglelefteq H$ & $H\cap K' \leq H'$ we know by Lemma §22.1 $(H\cap K')H' \leq H$.

Similarly $H' \trianglelefteq H$ & $H\cap K \leq H$ $\Rightarrow$ $(H\cap K)H' \leq H$.

We have $H\cap K' \subseteq H\cap K$ so $(H\cap K')H' \subseteq (H\cap K)H'$.

Claim 1: $K' \trianglelefteq K \Rightarrow H\cap K' \trianglelefteq H\cap K$

Pf/ $g \in H\cap K$, $h \in H\cap K'$ $\Rightarrow$ • $ghg^{-1} \in H$ $(g,h\in H)$

• $ghg^{-1} \in gK'g^{-1} = K'$ because $g\in K$

Thus $g\,H\cap K'g^{-1} = H\cap K'$ $\forall g \in H\cap K$

Claim 2: $H' \trianglelefteq H$ & $K' \trianglelefteq K$ $\Rightarrow$ $(H\cap K')H' \trianglelefteq (H\cap K)H$

Pf/ $g \in H\cap K$, $h\in H$, $x \in H\cap K'$, $y\in H'$ $\Rightarrow$ $$(gh)(xy)(gh)^{-1} = gh(xy)h^{-1}g^{-1} = \underbrace{\overset{\in H\cap K}{\overbrace{g}}\,h\,\underset{\in H\cap K'}{x}\,(gh)^{-1}}_{\in H\cap K' \text{ by Claim 1}}\;\underbrace{gh\,y\,(\underset{\in H}{gh})^{-1}}_{\in H'}$$

• By symmetry $(K\cap H')K' \trianglelefteq (K\cap H)K'$

- Next $H\cap K' \trianglelefteq H\cap K$ by Claim 1, $H'\cap K \leq H\cap K$, so $(H\cap K')(H'\cap K) \leq H\cap K$ by Lemma §22.1.

<u>Claim 3:</u> $H\cap K' \trianglelefteq H\cap K$ & $K\cap H' \trianglelefteq H\cap K \Rightarrow (H\cap K')(K\cap H') \trianglelefteq H\cap K$

Pf/ $x\in H\cap K'$, $y\in K'\cap H$, $h\in H\cap K \Rightarrow h x y h^{-1} = (hxh^{-1})(hyh^{-1})$

$x\in H\cap K'$, $h\in H\cap K \Rightarrow hxh^{-1}\in H\cap K'$ because $H\cap K' \trianglelefteq H\cap K$

$y\in K\cap H'$, $h^{-1}\in H\cap K \Rightarrow h^{-1}yh \in K\cap H'$ ——— $K'\cap H \trianglelefteq H\cap K$

So $hxyh^{-1} \in (H\cap K')(K\cap H')$ □

(2) By symmetry, it is enough to show $\dfrac{(H\cap K)H'}{(H\cap K')H'} \simeq \dfrac{H\cap K}{(H'\cap K)(K'\cap H)}$

This is precisely the 3rd Isomorphism theorem applied to the group $\tilde{G} := (H\cap K)H$ the subgroup $\tilde{H} := H\cap K$ & the normal subgroup $N := (H\cap K')H'$

<u>Claim 4:</u> $\tilde{H} N = (H\cap K)(H\cap K')H' = (H\cap K)H'$

Pf/ $H\cap K' \leq H\cap K$, so $(H\cap K)(H\cap K') = H\cap K$

<u>Claim 5:</u> $\tilde{H}\cap N := (H\cap K)\cap N = (H'\cap K)(K'\cap H)$

Pf/ $(H\cap K)\cap N = (H\cap K)\cap((H\cap K')H') \overset{?}{=} (H'\cap K)(K'\cap H)$

We know $(H'\cap K)(K'\cap H) = (K'\cap H)(H'\cap K)$ by Lemma §22.1 since $H'\cap K \trianglelefteq H\cap K$ & $K'\cap H \leq H\cap K$.

($\supseteq$) $H'\cap K \leq H\cap K$ & $K'\cap H \leq H\cap K \Rightarrow (H'\cap K)(K'\cap H) \leq H\cap K$

$H'\cap K \leq H' \Rightarrow (H\cap K')(K\cap H') \leq (H\cap K')H'$

($\subseteq$) Let $x\in (H\cap K)\cap((H\cap K')\cdot H')$ so $x = ab$ with $a\in H\cap K'$ & $b\in H'$

In particular $x\in H\cap K$ & $a\in H\cap K' \leq H\cap K \Rightarrow b = a^{-1}x \in (H\cap K)\cap H' = H'\cap K$

$\Rightarrow x = ab \in (H\cap K')(H'\cap K)$ as we wanted. □

By the 3rd Iso theorem combined with Claims 4 & 5, we get

$$\frac{(H\cap K)H'}{(H\cap K')H'} \underset{\text{Claim 4}}{=} \frac{\tilde{H}N}{N} \underset{\text{3rd Iso Theorem}}{\simeq} \frac{\tilde{H}}{\tilde{H}\cap N} \underset{\text{Claim 5}}{=} \frac{H\cap K}{(H'\cap K)(K'\cap H)}$$

← this is symmetric in (H, H') & (K, K')