Lecture XXXI: Solvable Groups II

Recall: We defined solvable groups as follows:

$G = G^{(0)}$; $G^{(1)} = [G^{(0)}, G^{(0)}]$, $G^{(2)} = [G^{(1)}, G^{(1)}]$, ... , $G^{(\ell+1)} = [G^{(\ell)}, G^{(\ell)}]$

By Lemma §30.2 $G^{(\ell+1)} \trianglelefteq G^{(\ell)}$

G is solvable if $G^{(N)} = \{e\}$ for sufficiently large N.

Q: How can we decide if G is solvable without computing all $G^{(\ell)}$?

TODAY'S GOAL: Find alternative tests for solvability

§31.1 Properties of Commutator subgroups:

Lemma: Fix G a group

(1) $G/[G,G]$ is abelian

(2) Any subgroup containing $[G,G]$ is normal.

(3) If $H \trianglelefteq G$ satisfies G/H is abelian, then $[G,G] \subseteq H$.

Proof: (1) was done in Lemma §30.2

(2) If $[G,G] \subseteq H \leq G$, then $\pi(H) \trianglelefteq G/[G,G]$ (because $G/[G,G]$ is abelian).
where $\pi : G \twoheadrightarrow G/[G,G]$ is the natural projection.

But $\ker(\pi) = [G,G] \subseteq H$, so $\pi^{-1}(\pi(H)) = H$.

By the 2nd Isomorphism Theorem $\left\{\begin{array}{c}\text{Normal subgroups}\\ \text{of } G/[G,G]\end{array}\right\} \xleftrightarrow{1-6-1} \left\{\begin{array}{c}\text{Normal subgroups}\\ \text{of } G \text{ containing}\\ [G,G]\end{array}\right\}$

$\pi(H) \longmapsto H = \pi^{-1}(\pi(H))$

(3) If $\pi : G \to G/H$, then $\pi(aba^{-1}b^{-1}) = \pi(a)\pi(b)\pi(a)^{-1}\pi(b)^{-1} = \{e\}$ because G/H is abelian. Thus, $[a,b] \in \ker \pi = H$ $\forall a,b \in G$, so $[G:G] \subseteq H$ □

Consequence: $G^{(\ell)}/G^{(\ell+1)}$ is abelian $\forall \ell \geq 0$

The next result gives a converse to this assertion.

Theorem: Let G be a group. Then, G is solvable if, and only if, there exists a composition series $\Sigma : G = K_0 \trianglerighteq K_1 \trianglerighteq \cdots \trianglerighteq K_m = \{e\}$ such that $\mathfrak{gr}_j^{\Sigma}(G) = K_j/K_{j+1}$ is abelian for every $j = 0, 1, \ldots, m$.

Proof: ($\Rightarrow$) Assume G is solvable, so $G^{(N)} = \{e\}$ for some N. Take $K_j = G^{(j)}$ for all j & $m = N$. This descending sequence is a composition series

($\Leftarrow$) We will prove that $K_j \supseteq G^{(j)}$ $\forall j = 0, 1, \ldots, m$ by induction on j

Base case: $j=0$ $\quad G^{(0)} = G = K_0$

Induction Step: Fix $0 \le j < m$ & assume $G^{(j)} \subseteq K_j$

Since $K_{j+1} \trianglelefteq K_j$ & K_j/K_{j+1} is abelian, by Lemma (2), we have $[K_j : K_j] \subseteq K_{j+1}$

Then $G^{(j+1)} = [G^{(j)}, G^{(j)}] \underset{\text{(IH)}}{\subseteq} [K_j, K_j] \subseteq K_{j+1}$

In particular, $G^{(m)} \subseteq K_m = \{e\}$, so $G^{(m)} = \{e\}$. Thus, G is solvable.

§31.2 Consequences of Theorem §31.1:

Corollary 1: Let G be a group and $H \le G$ a subgroup. If G is solvable, so is H.

Proof: Assume G is solvable, and let $n \ge 0$ be so that $G^{(n)} = \{e\}$.

- We build the commutator series for H.

Claim: $H^{(\ell)} \subseteq G^{(\ell)}$ $\forall \ell$

Pf/ Easy induction on ℓ $\quad H \subseteq G \Rightarrow H^{(1)} = [H, H] \subseteq [G, G] = G^{(1)}$, etc.

In particular, $H^{(n)} \subseteq G^{(n)} = \{e\}$ so H is solvable.

Corollary 2 (Criteria for solvability):

Let G be a group and $N \trianglelefteq G$ a normal subgroup. Then

G is solvable if, and only if, (N is solvable & G/N is solvable)

Proof: ($\Rightarrow$) Assume G is solvable and pick $n \ge 0$ with $G^{(n)} = \{e\}$. By Corollary 1, N is solvable (& $N^{(n)} = \{e\}$)

Now, we consider the natural projection $\pi : G \twoheadrightarrow G/N$.

Claim: $(G/N)^{(\ell)} = \pi(G^{(\ell)})$ $\quad \forall \ell \ge 0$

Pf/ By induction on ℓ.

Base case: $\ell = 0$ $\quad (G/N)^{(0)} = G/N = \pi(G^{(0)})$ ✓

Induction Step: We assume $\ell \geq 0$ & $(G/N)^{(\ell)} = \pi(G^{(\ell)})$.

Since π is a group homomorphism, we have

$$\pi([G^{(\ell)}, G^{(\ell)}]) = [\pi(G^{(\ell)}), \pi(G^{(\ell)})] = [(G/N)^{(\ell)}, (G/N)^{(\ell)}] = (G/N)^{(\ell+1)}$$

In particular $(G/N)^{(n)} = \pi(G^{(n)}) = \pi(e) = eN$ is unit of G/N. Hence, G/N is solvable.

($\Leftarrow$) We assume N & G/N are solvable. By Theorem 31.1, there exist composition series

$$\Sigma_1 : \quad G/N = \overline{G}_0 \trianglerighteq \overline{G}_1 \trianglerighteq \overline{G}_2 \trianglerighteq \cdots \trianglerighteq \overline{G}_\ell = \{e\}$$

$$\Sigma_2 \quad N = N_0 \trianglerighteq N_1 \trianglerighteq N_2 \trianglerighteq \cdots \trianglerighteq N_k = \{e\}$$

with abelian graded pieces.

Using the natural projection $\pi : G \longrightarrow G/N$, we define $\begin{cases} G_j := \pi^{-1}(\overline{G}_j) \leq G & 0 \leq j \leq \ell \\ G_{\ell+i} = N_i & 0 \leq i \leq k \end{cases}$

to get $\Sigma : \quad G = G_0 \geq G_1 \geq \cdots \geq G_\ell = \underbrace{N \trianglerighteq G_{\ell+1} \trianglerighteq \cdots \trianglerighteq G_{\ell+k} = \{e\}}_{\text{graded pieces here are abelian}}$

For the first part of Σ $(G \geq G_1 \geq \cdots \geq G_\ell)$, we need to show

(i) $G_j \trianglerighteq G_{j+1} \quad \forall j = 0, \ldots, \ell-1$

(ii) $G_j/G_{j+1} \simeq \overline{G}_j/\overline{G}_{j+1}$ (hence abelian)

If this where true, G would have a composition series (Σ) with abelian graded piece, and the Theorem would imply that G is solvable.

Proof of (i) and (ii):

$$\begin{tikzcd} G \arrow[r, two heads, "\pi"] & G/N \\ G_j \arrow[u, phantom, sloped, "\leq"] \arrow[r, "\pi_j"] & \overline{G}_j \arrow[u, phantom, sloped, "\leq"] \end{tikzcd}$$

π_j is surjective by definition.

$$\begin{tikzcd} G_j \arrow[r, two heads, "\pi_j"] \arrow[rr, bend right, "f_j"'] & \overline{G}_j \arrow[r, two heads] & \overline{G}_j/\overline{G}_{j+1} \end{tikzcd}$$

Then f_j is a surjective group homomorphism.

$$\mathrm{Ker}(f_j) = \pi_j^{-1}(\overline{G}_{j+1}) = \pi^{-1}(\overline{G}_{j+1}) \cap G_j = G_{j+1} \cap G_j = G_{j+1}$$

Hence, $\underset{(i)}{G_{j+1} \trianglelefteq G_j}$ & $G_j/G_{j+1} \underset{(ii)}{\simeq} \overline{G}_j/\overline{G}_{j+1}$ by the 1st Iso Theorem □

Corollary 3: Every p-group is solvable.

Recall from Theorem §14.2, that p-groups have non-trivial centers. We obtained this from a more general statement

$$\boxed{G \circlearrowright X \quad \& \quad |G| = p^r \quad r \geq 1 \quad \Longrightarrow \quad |X| \equiv |\underbrace{X^G}_{\{x \,|\, g\cdot x = x \ \forall g\in G\}}| \pmod p}$$

Proof: X = disjoint union of orbits under G-action

$\#$ elements in an orbit divides $|G| = p^r$ by Stabilizer-Orbit correspondence

$\Rightarrow |X| \equiv \#$ orbits that have just one element $\pmod p$

$\equiv |X^G| \mod (p)$

In particular, if we let $X = G$ & $G \circlearrowright$ by conjugation, then

$$|G| \equiv |\{x \in G : gxg^{-1} = x \ \forall g \in G\}| = |Z(G)| \pmod p$$

Proof of Corollary 3: By Proposition §14.2, G admits a chain of normal subgroups

$$\{e\} = Z_0 \underset{\neq}{\trianglelefteq} Z_1 \underset{\neq}{\trianglelefteq} Z_2 \underset{\neq}{\trianglelefteq} \cdots \underset{\neq}{\trianglelefteq} Z_s = G$$

such that each quotient Z_{i+1}/Z_i $(1 \leq i \leq s-1)$ is an abelian p-group

In the proof, we picked $Z_1 = Z(G)$. & proved the statement by induction on $r \geq 1$ if $|G| = p^r$.

Setting $K_i = Z_{s-i}$ we have

$$\Sigma: \quad G = K_0 \underset{\neq}{\trianglerighteq} K_1 \underset{\neq}{\trianglerighteq} \cdots \underset{\neq}{\trianglerighteq} K_{s-1} \underset{\neq}{\trianglerighteq} K_s = \{e\}$$

with K_i/K_{i+1} abelian $\forall i = 0, \ldots, s-1$. Thus, by Theorem §31.2, G is solvable.