Lecture XXXII: Nilpotent Groups I

Summary: Main properties of solvable groups.

(1) $G^{(n)} = \{e\}$ for some $n \geq 0$, where $G^{(0)} = G$ $\quad G^{(\ell+1)} = [G^{(\ell)}, G^{(\ell)}]$ $\forall \ell \geq 0$

(2) There exists a composition series of G with abelian graded pieces

(3) Sub & quotients of solvable groups are solvable

(4) $N \trianglelefteq G$ & G/N are solvable $\Rightarrow$ G is solvable.

§32.1 Jordan-Hölder series and solvable groups:

Proposition: Let G be a finite, solvable group. Then, for any Jordan-Hölder series

$$\Sigma : G = H_0 \trianglerighteq H_1 \trianglerighteq H_2 \trianglerighteq \cdots \trianglerighteq H_\ell = \{e\}$$

of G, the associated graded pieces are abelian and simple.

Hence $H_j/H_{j+1} \simeq \mathbb{Z}/p_j\mathbb{Z}$ for $p_0, p_1, \ldots, p_{\ell-1}$ primes.

Proof: We know that graded pieces of Jordan-Hölder series are simple (Lemma §28.4)

Since G is solvable, G admits a composition series Σ_1 with abelian graded pieces.

Let $\Sigma_1 : G = K_0 \trianglerighteq K_1 \trianglerighteq K_2 \trianglerighteq \cdots \trianglerighteq K_m = \{e\}$ be this series.

Now, by Schreier's Theorem we can find a common refinement between Σ & Σ_1 up to equivalence. Let Σ' be such refinement.

- However, Σ is Jordan-Hölder, any refinement of Σ is trivial. That is, we can only have a silly refinement

$$\Sigma' : G = H_0 = \cdots = H_0 \underset{\neq}{\trianglerighteq} H_1 = H_1 = \cdots = H_1 \underset{\neq}{\trianglerighteq} H_2 \cdots \underset{\neq}{\trianglerighteq} H_\ell = H_\ell = \cdots = H_\ell = \{e\}.$$

In particular, the non-trivial graded pieces of Σ' are simple.

Claim: Since the graded pieces of Σ_1 are abelian, any refinement of Σ_1 will have abelian graded pieces.

Proof: Any refinement between $K_i \trianglerighteq K_{i+1}$ will give $K_i \trianglerighteq K_{i,1} \trianglerighteq \cdots \trianglerighteq K_{i,s_i} = K_{i+1}$

and it corresponds to a refinement of $K_i/K_{i+1} \trianglerighteq \{e\}$ via $K_i/K_{i+1} \trianglerighteq \pi(K_{i,1}) \trianglerighteq \cdots \trianglerighteq \pi(K_{i,s_i}) = \{e\}$

where $\pi : K_i \twoheadrightarrow K_i/K_{i+1}$ is the natural projection.

Since K_i/K_{i+1} is abelian, then $\pi(K_{i,j})$ is abelian for all j, So $\frac{\pi(K_{i,j})}{\pi(K_{i,j+1})}$ is abelian $\forall j$

Furthermore $\frac{\pi(K_{i,j})}{\pi(K_{i,j+1})} = \frac{K_{i,j}/K_{i+1}}{K_{i,j+1}/K_{i+1}} \simeq K_{i,j}/K_{i,j+1}$

$K_{i+1} \trianglelefteq K_i$

$K_{i+1} \subseteq K_{i,j} \subseteq K_i$

$K_{i+1} \subseteq K_{i,j+1} \subseteq K_i$

3rd Iso Thm applied to

$K_{ij} \twoheadrightarrow K_{ij}/K_{i+1}$

$K_{i,j+1} \trianglelefteq K_{i,j}$

$\Rightarrow K_{i,j}/K_{i,j+1}$ is abelian $\forall j$, as we wanted to show.

Conclusion: Σ' has graded pieces that are abelian & either trivial or simple.

Thus, the graded pieces of Σ, which are the non-trivial pieces of Σ' are abelian and simple, as we wanted to show.

. To finite, we use the 2^{nd} Structure Thm applied to the graded pieces of Σ', which are finite, abelian groups. Indeed $gr_j^{\Sigma}(G) \simeq \mathbb{Z}/d_1^{(j)}\mathbb{Z} \times \cdots \times \mathbb{Z}/d_r^{(j)}\mathbb{Z}$ with $d_i^{(j)} | d_{i+1}^{(j)}$ $\forall i$

Since these groups are simple, we must have $r=1$.

Now, $\mathbb{Z}/d\mathbb{Z}$ is simple if, and only if, d is prime.

Therefore $gr_j^{\Sigma}(G) \simeq \mathbb{Z}/p_j\mathbb{Z}$ for some prime p_j. This concludes the proof. $\square$

§32.2 (Lower) Central Series:

Definition: Let G be a group. Define

$C^1(G) = G$, $C^2(G) = [G, C^1(G)] \trianglelefteq G$ (by Lemma §30.1)

$C^3(G) = [G, C^2(G)] \trianglelefteq G$ (by Lemma §30.1)

$\cdots$ $C^{j+1}(G) = [G, C^j(G)] \trianglelefteq G$ (by Lemma §30.1 since $G \trianglelefteq G$, $C^j(G) \trianglelefteq G$)

We call $C^1(G) \trianglerighteq C^2(G) \trianglerighteq \cdots \trianglerighteq C^j(G) \trianglerighteq C^{j+1}(G) \trianglerighteq \cdots$ the (lower) central series of G

⚠ Need to check $C^k(G) \trianglerighteq C^{k+1}(G)$ $\forall k$ (see Remark (2) below)

Definition: A group G is nilpotent if $C^n(G) = \{e\}$ for some $n \geq 1$.

Remarks: (1) $C^2(G) = [G, C^1(G)] = [G,G] = G^{(1)}$ (for the commutator series)

Corollary 1: G abelian $\Rightarrow$ G nilpotent.

(2) $C^{k+1}(G) = [G, C^k(G)]$ is generated by $\{ gxg^{-1}x^{-1} : g \in G,\ x \in C^{(k)}(G)\}$

But $C^k(G) \trianglelefteq G \implies gxg^{-1} \in C^k(G)$. Since $x^{-1} \in C^k(G)$, we get $gxg^{-1}x^{-1} \in C^k(G)$.

This is why $C^k(G) \trianglelefteq G$ $\forall k$. In particular, $C^{k+1}(G) \subseteq C^k(G)$, so $C^{k+1}(G) \trianglelefteq C^k(G)$.

(3) For a nilpotent group G, the central series of G is a composition series.

$$G = C^1(G) \trianglerighteq C^2(G) \trianglerighteq \cdots \trianglerighteq C^m(G) = \{e\} \qquad (\text{for some } m)$$

Moreover, it has the property that

$$C^{n+1}(G) = [G, C^n(G)] \supseteq [C^n(G), C^n(G)]$$

Hence, by Lemma §30.1, $\dfrac{C^n(G)}{C^{n+1}(G)}$ is abelian. (By the 3rd isomorphism theorem

we have $\dfrac{C^n(G)}{C^{n+1}(G)} \simeq \dfrac{\dfrac{C^n(G)}{[C^n(G), C^n(G)]}}{\dfrac{C^{n+1}(G)}{[C^n(G), C^n(G)]}}$. This, combined with the fact that quotients of abelian

groups are abelian, proves the claim).

<u>Corollary 2</u>: G nilpotent $\implies$ G solvable.

⚠ The converse is not true!

<u>Example</u>: $B = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \ a, c \neq 0 \ b \in \mathbb{C} \right\} \leq GL_2(\mathbb{C})$ (invertible upper triangular matrices)

$$\begin{aligned}
\implies C^1(B) &= B \\
&\ \trianglerighteq \\
C^2(B) &= [B, B] = \left\{ \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \ x \text{ arbitrary} \right\} \qquad (\text{§30.3 Example 2}) \\
&\ \trianglerighteq \\
C^3(B) &= [B, C^2(B)]
\end{aligned}$$

A typical generator of $C^3(B)$ is of the form:

$$\begin{bmatrix} a_1 & b_1 \\ 0 & c_1 \end{bmatrix} \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{a_1} & \frac{-b_1}{a_1 c_1} \\ 0 & \frac{1}{c_1} \end{bmatrix} \begin{bmatrix} 1 & -x \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a_1 & a_1 x + b_1 \\ 0 & c_1 \end{bmatrix} \begin{bmatrix} 1/a_1 & -\frac{x}{a_1} - \frac{b_1}{a_1 c_1} \\ 0 & 1/c_1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \qquad \text{where } d = a_1\left(-\frac{x}{a_1} - \frac{b_1}{a_1 c_1}\right) + \frac{1}{c_1}(a_1 x + b_1)$$

$$= \frac{-a_1 c_1 x - a_1 b_1 + a_1^2 x + a_1 b_1}{a_1 c_1} = x\left(\frac{a_1}{c_1} - 1\right)$$

Note: d need not be 0 (if $x \neq 0$ & $\frac{a_1}{c_1} \neq 1$)

In fact, d is arbitrary.

$\implies C^3(B) = C^2(B)$. By induction $C^2(B) = C^n(B)$ $\forall n \geq 2$.

Hence B is not nilpotent. However, we know it is solvable because $C^2(B)$ is abelian $\Rightarrow$ $B^{(2)} = [B^{(1)}, B^{(1)}] = [C^2(B), C^2(B)] = \{e\}$.

§ 32.3 Properties of Nilpotent groups:

The next result gives an alternative characterization of nilpotent groups.

Theorem: Let G be a group. Then, G is nilpotent if, and only if, G has a composition series

$$\Sigma: \quad G = H_0 \trianglerighteq H_1 \trianglerighteq H_2 \trianglerighteq \cdots \trianglerighteq H_m = \{e\}$$

such that $[G, H_\ell] \subseteq H_{\ell+1}$ for every $\ell = 0, 1, \dots, m-1$.

In particular, $[H_\ell, H_\ell] \subseteq H_{\ell+1}$, so $H_\ell / H_{\ell+1}$ is abelian $\forall \ell$

Proof: ($\Rightarrow$) Take Σ to be the lower central series, ie $H_\ell = C^{\ell+1}(G)$ $\forall \ell$

($\Leftarrow$) Assume such a composition series Σ exists.

Claim: $C^{\ell+1}(G) \subseteq H_\ell$ $\forall \ell = 0, \dots, m$.

Pf/ By induction on ℓ.

Base Case: $\ell = 0$ $\quad C^1(G) = G = H_0$

Inductive Step: $C^{\ell+2}(G) = [G, C^{\ell+1}(G)] \underset{\text{by (IH)}}{\subseteq} [G, H_\ell] \subseteq H_{\ell+1}$ by assumption

$\square$

Since $H_m = \{e\}$, we conclude that $C^{m+1}(G) = \{e\}$, so G is nilpotent.

Next, we discuss more properties of nilpotent groups, analogous to those for solvable groups discussed in Lecture 31.

Corollary 1: Let G be a group and $H \leq G$ a subgroup. If G is nilpotent, so is H.

Proof: Enough to show: $C^\ell(H) \subseteq C^\ell(G)$ $\forall \ell = 1, \dots$ By induction on ℓ

Base Case: $\ell = 1$: $C^1(H) = H \subseteq G = C^1(G)$

Inductive Step: Assume $C^\ell(H) \leq C^\ell(G)$

$$C^{\ell+1}(H) = [H, C^\ell(H)] \underset{H \leq G \ \& \ (IH)}{\subseteq} [G, C^\ell(G)] = C^{\ell+1}(G).$$

Since G is nilpotent, then $\exists\, m$ st $C^m(G) = \{e\}$. Thus $C^m(H) = \{e\}$, showing H is also nilpotent.

Corollary 2 Let G be a group and $N \trianglelefteq G$ a normal subgroup. If G is nilpotent, then both N and G/N are also nilpotent.

Proof: By Corollary 1 we know N is nilpotent. To show G/N is nilpotent, it is enough to show $C^\ell(G/N) = \pi(C^\ell(G))$ $\forall \ell = 1, 2, \dots$ where $\pi : G \longrightarrow G/N$ is the natural projection. We do so by induction on ℓ

Base Case: $\ell = 1$: $C^1(G/N) = G/N = \pi(G) = \pi(C^1(G))$

Inductive Step: Assume $C^\ell(G/N) = \pi(C^\ell(G))$

$$C^{\ell+1}(G/N) = [G/N, C^\ell(G/N)] = [\pi(G), \pi(C^\ell(G))] \underset{\pi \text{ gp hom}}{=} \pi([G, C^\ell(G)]) = \pi(C^{\ell+1}(G)).$$

If G is nilpotent, $\exists\, m$ st $C^m(G)$. Thus $C^m(G/N) = \pi(C^m(G)) = \pi(\{e\}) = \{eN\}$. So G/N is nilpotent as well. $\square$

The next result indicates the converse to Corollary 2 is not true (N needs to satisfy an extra condition)

Theorem (Criteria for nilpotency):

Let G be a group. Then G is nilpotent if, and only if, there exists $A \trianglelefteq G$ with $A \leq Z(G)$ (center of G) such that G/A is nilpotent

Note: $A \leq Z(G)$ is abelian, so it is automatically nilpotent. In addition, $A \leq Z(G) \Rightarrow A \trianglelefteq G$.

Proof: $(\Longrightarrow)$ Follows from Corollary 2 (Take $N = Z(G)$)

$(\Longleftarrow)$ Consider the natural projection $\pi : G \longrightarrow G/A$. Let $n \geq 1$ be such that

$$C^n(G/A) = \{e_{G/A}\} \qquad (n \text{ exists because } G/A \text{ is nilpotent})$$

By Proof of Corollary 2, we know $C^n(G/A) = \pi(C^n(G)) = \{e_{G/A}\}$, so $C^n(G) \subseteq \ker \pi = A$. Since $A \leq Z(G)$ we have

$$C^{n+1}(G) = [G, C^n(G)] \leq [G, A] = \{e_G\}, \text{ so } C^{n+1}(G) = \{e_G\}.$$

We conclude from this that G is nilpotent.

Corollary 3: All p-groups are nilpotent.

Proof: Let G be a p-group, $|G| = p^r$, $r \geq 1$. We prove the statement by complete induction on r.

Base case: $r = 1$. Then G is cyclic, hence abelian. By Corollary 1 §32.2, G is nilpotent.

Inductive Step: Let $A = Z(G)$. By Theorem §14.2, $Z(G) \neq \{e\}$.

If $Z(G) = G$, then G is nilpotent because it is abelian.

Otherwise, $\{e\} \subsetneq Z(G) \subsetneq G$. Then G/A is a p-group of order p^ℓ with $\ell < r$.

By (IH) G/A is nilpotent. Since $A \leq Z(G)$, Theorem implies G is nilpotent. $\square$