Lecture XXXIII: Nilpotent Groups II

Recall: Properties of solvable and nilpotent groups:

(1) Nilpotent $\Rightarrow$ Solvable (NOT conversely)

(2) Abelian $\Rightarrow$ Nilpotent

$|G| = p^n$ $n \geq 1$ p prime $\Rightarrow$ Nilpotent

(3) Sub- & quotient groups of a solvable (respectively, nilpotent) group are solvable (respectively, nilpotent).

(4) $N \trianglelefteq G$ & G/N solvable $\Rightarrow$ G solvable

(5) $A \leq Z(G)$ & G/A nilpotent $\Rightarrow$ G nilpotent.

TODAY'S GOAL: Characterize finite Nilpotent groups.

Main Theorem: Any finite nilpotent group G is a direct product of p-groups (the primes can vary). These p-groups are Sylow p-subgroups of G.

§33.1 A useful lemma:

Lemma: Let G be a group and $K \leq G$ subgroup. If $[G:K] \subseteq K$, then $K \trianglelefteq G$.

Proof: $\forall g \in G, k \in K$ $\quad gkg^{-1}k^{-1} \in [G:K] \leq K \Rightarrow gkg^{-1}k^{-1} \in K$

Thus, $gkg^{-1} \in K$, so $gKg^{-1} \in K$ $\forall g \in G$. This implies $K \trianglelefteq G$.

Definition: Let G be a group and $H \leq G$ a subgroup. The normalizer of H in G is $N_G(H) := \{g \in G : gHg^{-1} = H\}$

Remark: $N_G(H)$ is a subgroup of G & $H \subseteq N_G(H)$

Useful Lemma: Let G be a nilpotent group and $H \lneq G$ a proper subgroup. Then, we have $H \subsetneq N_G H$.

Proof: As G is nilpotent, we have a composition series

$\Sigma: \quad G = K_0 \trianglerighteq K_1 \trianglerighteq K_2 \trianglerighteq \cdots \trianglerighteq K_m = \{e\}$

such that $[G:K_\ell] \subseteq K_{\ell+1}$ for $\ell = 0, 1, \ldots, n-1$ (Example: Σ lower central series)

By the Lemma $[G:K_\ell] \subseteq K_{\ell+1} \subseteq K_\ell \Rightarrow K_\ell \trianglelefteq G$.

Let G_ℓ be the subgroup of G generated by H & K_ℓ. By Lemma §22.1

$$G_\ell = H\cdot K_\ell = K_\ell \cdot H$$

<u>Claim:</u> $G_{\ell+1} \trianglelefteq G_\ell$ for all $\ell = 0, \ldots, m-1$

Assume the claim is true. This gives:

$$G_0 = G \trianglerighteq G_1 \trianglerighteq G_2 \trianglerighteq \cdots \trianglerighteq G_m = H$$

Since $H \subsetneq G$ $\exists$ $k \in \{0, \ldots, m-1\}$ with $G_k \ntrianglerighteq G_{k+1} = \cdots = G_m = H$

Then, $H \ntrianglelefteq G_k$ implies $G_k \subseteq N_G(H)$. Hence $H \subsetneq N_G(H)$, as we wanted □

The claim in the proof of the Useful Lemma is a general fact that we now prove.

<u>Proposition:</u> Let G be a group, $N_2 \subseteq N_1$ two normal subgroups of G with $[G:N_1] \subseteq N_2$. Let $H \leq G$ be a subgroup. Then, $N_2 \cdot H$ is normal in $N_1 \cdot H$.

<u>Proof:</u> Since $N_i \trianglelefteq G$ for $i = 1, 2$ Lemma §22.1 implies $N_i H \leq G$ is a subgroup.

We need to show $\forall\, a \in N_1 H$ & $b \in N_2 H$ $aba^{-1} \in N_2 H$.

It suffices to treat 2 cases, namely when $a \in N_1$ & when $a \in H$.

<u>CASE 1:</u> $a \in H \subseteq N_1 \cdot H$

If $b = n_2 x$ with $n_2 \in N_2$, $x \in H$ then $aba^{-1} = a n_2 x a^{-1} = \underbrace{a n_2 a^{-1}}_{\substack{\in N_2 \\ (N_2 \trianglelefteq G)}} \underbrace{a x a^{-1}}_{\substack{\in H \\ (a, x \in H)}} \in N_2 H$.

<u>CASE 2:</u> $a \in N_1 \subseteq N_1 \cdot H$

If $b \in N_2 H$, then $[b^{-1}:a] = b^{-1} a b a^{-1} \in [G:N_1] \subseteq N_2 \Rightarrow aba^{-1} \in b N_2 \subseteq N_2 H$

$N_2 H = H N_2$
$b \in N_2 H$

In both cases we have $aba^{-1} \in N_2 H$, so $N_2 H \trianglelefteq N_1 H$ because $N_1 H$ is generated by N_1 & H. □

<u>§ 33.2 Self-normalizing lemma for Sylow subgroups:</u>

<u>Lemma:</u> Let G be a finite group & p a prime dividing $|G|$. Let $P \in Syl_p(G)$. Let $L \leq G$ be a subgroup. If $N_G(P) \leq L \leq G$, then $N_G(L) = L$.

Proof: We have $P \leq N_G(P) \leq L$ by assumption. In particular, $P \in Syl_p(L)$ since maximal power of p dividing $|L|$ is $|P|$ because $P \subseteq L$.

Let $g \in N_G(L)$. Then, $P \subseteq L$ & $gPg^{-1} \subseteq gLg^{-1} = L$ so $P, gPg^{-1} \in Syl_p(L)$
In particular, by Sylow Thm (2), $\exists\, \ell \in L$ with $P = \ell(gPg^{-1})\ell^{-1} = (\ell g)P(\ell g)^{-1}$
This means $\ell g \in N_G(P) \subseteq L$, so $g \in \ell^{-1}L = L$.
$\ell \in L$

We conclude $N_G(L) \subseteq L$. Since $L \subseteq N_G(L)$, we get equality □

§33.3 Main Theorem:

Theorem: Let G be a finite group. The following conditions are equivalent.

(1) G is nilpotent
(2) Every Sylow p-subgroup is normal.
(3) G is isomorphic to a direct product of p-groups.

Proof: $(1) \Rightarrow (2)$: We assume G is nilpotent & fix $P \in Syl_p(G)$ for p prime with $p \mid |G|$. Let $H = N_G(P)$. We have two possibility:

- If $H = G$, then $P \trianglelefteq G$ and we are done.
- If $H \subsetneq G$, then:

  (i) $H \subsetneq N_G(H)$ by Useful Lemma §33.1
  (ii) $P \subseteq N_p(G) \subseteq H \subseteq G \Rightarrow N_G(H) = H$ by Lemma §33.2

  } Contradiction!

$(2) \Rightarrow (3)$: If $|G| = p_1^{a_1} \cdots p_k^{a_k}$ write $Syl_{p_i}(G) = \{P_i\}$ for $i = 1, \ldots, k$ ($P_i \trianglelefteq G$ by (2))
Then, $P_1 \times \cdots \times P_k \xrightarrow{\varphi} G$ is a group isomorphism.

$$(x_1, \ldots, x_k) \longmapsto x_1 \cdots x_k$$

Indeed $P_i \trianglelefteq G$, $P_j \trianglelefteq G$ & $P_i \cap P_j = \{e\}$ if $i \neq j$ for size reasons.
Lemma §19.1 $\Rightarrow ab = ba$ $\forall a \in P_i, b \in P_j$ with $i \neq j$ (P_i & P_j commute)
Thus, $\varphi(\underline{x})\varphi(\underline{y}) = \varphi(\underline{x}\cdot\underline{y})$ $\forall\, \underline{x}, \underline{y} \in \prod_{j=1}^{k} P_j$. φ is surjective for size reasons $\Rightarrow \varphi$ iso.

$(3) \Rightarrow (1)$: Assume $G \simeq P_1 \times P_2 \times \cdots \times P_\ell$ where $|P_j| = p_j^{n_j}$ for some primes $p_1, \ldots, p_\ell$ (all distinct) and $n_1, \ldots, n_\ell \in \mathbb{Z}_{\geq 1}$.

Recall that p-groups are nilpotent, so we know all $\mathfrak{P}_i$'s are nilpotent.
Thus, To prove the statement we must show that direct product of nilpotent groups is nilpotent. It's enough To prove it for $\ell = 2$. The general case follows from this one, combined with an easy induction. We do this case in a separate Proposition $\square$

Proposition: Let G_1 & G_2 be nilpotent groups. Then, $G_1 \times G_2$ is also nilpotent.

Proof: Since G_1 & G_2 are nilpotent, we can find 2 composition series:

$$\Sigma_1 : G_1 = H_0 \trianglerighteq H_1 \trianglerighteq \cdots \trianglerighteq H_s = \{e\} \quad \text{with} \quad [G : H_\ell] \subseteq H_{\ell+1} \quad \forall \ell = 0, 1, \ldots, s-1.$$

$$\Sigma_2 : G_2 = K_0 \trianglerighteq K_1 \trianglerighteq \cdots \trianglerighteq K_t = \{e\} \quad \text{with} \quad [G_1 : K_\ell] \subseteq K_{\ell+1} \quad \forall \ell = 0, \ldots, t-1$$

Take $\Sigma : G_1 \times G_2 = L_0 \trianglerighteq \underbrace{G_1 \times K_1}_{L_1} \trianglerighteq \underbrace{G_1 \times K_2}_{L_2} \trianglerighteq \cdots \trianglerighteq \underbrace{G_1 \times K_t}_{L_t} \simeq G_1 \trianglerighteq \underbrace{H_1}_{L_{t+1}} \trianglerighteq \underbrace{H_2}_{L_{t+2}} \trianglerighteq \cdots \trianglerighteq \underbrace{H_s}_{L_{t+s}} = \{e\}$

ie $L_\ell = G_1 \times K_\ell \trianglelefteq G_1 \times G_2 \quad \forall \ell = 0, \ldots, t$ since $K_\ell \trianglelefteq G_2$ by Lemma § 33.1

$L_{t+j} = H_j \times \{e_2\} \subseteq G_1 \times G_2 \quad \forall j = 0, \ldots, s$

Claim: $[G_1 \times G_2, L_\ell] \subseteq L_{\ell+1} \quad \forall \ell = 0, \ldots, t+s-1$

Pf/ If $0 \leq \ell \leq t-1$, then $[G_1 \times G_2, G_1 \times K_\ell] \subseteq G_1 \times [G_2, K_\ell] \subseteq G_1 \times K_{\ell+1} = L_{\ell+1}$

If $0 \leq j \leq s$, $[G_1 \times G_2 : L_{t+j}] = [G_1 : H_j] \times \{e_2\} \subseteq H_{j+1} \times \{e_2\} = L_{t+j+1}$

Conclusion, by Theorem § 32.3, $G_1 \times G_2$ is nilpotent. $\square$