Lecture XLI: Maximal ideals

Recall: Let R be a commutative ring and $I \subsetneq R$ a proper ideal.

I is prime $\iff$ R/I is an integral domain

$\iff \forall a,b \in R \; [\, ab \in I \implies a \in I$ or $b \in I \,]$

I is maximal $\iff$ R/I is a field

$\iff$ I is maximal among all proper ideals of R with respect to inclusion (ie $I \subseteq J \subseteq R$ ideal $\implies J = I$ or $J = R$.)

⚠ We still have to see if maximal ideals exist for any R commutative ring. In general this is achieved by using Zorn's Lemma (see §41.2). For a particular class of rings, called Noetherian rings, an alternate proof can be given which avoids the use of Zorn's Lemma. We will see this in a future lecture.

§41.1 The Geometry of commutative rings:

Geometrically, we can think of commutative rings as rings of functions valued in a field. To fix ideas, we assume this field is $\mathbb{C}$.

Heuristically X is a (topological) space $\rightsquigarrow$ $\mathrm{Fun}(X,\mathbb{C}) = \{ f: X \longrightarrow \mathbb{C}$ (continuous)$\}$

- it is a ring with pointwise $+$ and $\cdot$
- $\mathbb{0}$ = constant 0 function.
- $\mathbb{1}$ = ——— 1 ———

$Y \subseteq X$ (closed) subspace $\rightsquigarrow$ $I_Y = \{ f: X \longrightarrow \mathbb{C} \mid f_{(y)} = 0 \;\; \forall y \in Y \}$ ideal.

So ideals of $\mathrm{Fun}(X,\mathbb{C})$ correspond to closed subsets.

If X has an algebraic nature, then we restrict $\mathrm{Fun}(X,\mathbb{C})$ to polynomial / rational functions

Examples: $X = \mathbb{R}^2$ $\rightsquigarrow$ R = polynomial (real-valued) functions on X
$= \mathbb{R}[x,y]$

Q: What do we gain?

A: We can detect non-transversal intersections (ie multiplicities)

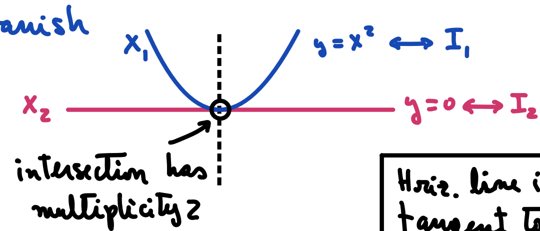
Example: $I_1 = (y - x^2) \subset R = \mathbb{R}[x,y]$

$I_2 = (y) \subset R = \mathbb{R}[x,y]$

Subsets of $\mathbb{R}^2$ where functions from I_1 (resp. I_2) vanish

$X_1 = \{ (a,b) \in \mathbb{R}^2 : b = a^2 \}$ (parabola)

$X_2 = \{ (a,0) \in \mathbb{R}^2 : a \in \mathbb{R} \}$ (x-axis)



Horiz. line is tangent to the parabola at (0,0)

Intersection of sets $X_1 \cap X_2 = \{(0,0)\}$ but we seem to have "lost" the multiplicity.

Right idea: define the ideal of functions vanishing at $X_1 \cap X_2$

$$I = I_1 + I_2 = (y - x^2, y) = (y, x^2)$$

This is NOT the ideal (x,y)

exponent 2 indicates we have multiplicity 2.

- Commutative ring R = [Type] functions on [Type] space X with values in $\mathbb{C}$ or some other fixed field.

[Type] functions	[Type] space
continuous	topological
linear	vector
polynomial	algebraic
⋮	⋮

Ideals = subsets of functions which vanish on a given subset $Y \subseteq X$

in the topological setting Y must be a closed set.

⇝ "Open sets" must then be given by non-vanishing of a set of functions

Eg: $GL_2(\mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2\times 2}(\mathbb{R}) \;\middle|\; ad - bc \neq 0 \right\}$

To learn more: Cox, Little, O'Shea: "Ideals, Varieties and Algorithms"

§41.2 Existence of Maximal Ideal:

Proposition 1: Let R be a commutative ring and $J \subsetneq R$ be a proper ideal. Then, $\exists\, M \subsetneq R$ a maximal ideal with $J \subseteq M$.

Proof: We prove the statement using Zorn's Lemma

Zorn's Lemma: Let $\mathcal{J} \neq \emptyset$ be a non-empty set & $\leq$ a partial order on $\mathcal{J}$.

(• Meaning of order: $i \leq i \quad \forall i \in \mathcal{J}$ [reflexive]

$\forall i,j \in \mathcal{J}$: $i \leq j$ & $j \leq i \Rightarrow i = j$ [antisymmetric]

$\forall i,j,l \in \mathcal{J}$: $i \leq j$ & $j \leq l \Rightarrow i \leq l$ [transitive]

• Partial : given $i, j \in \mathcal{J}$ it is possible that neither $i \le j$ nor $j \le i$ hold.

ie, not every pair of elements of $\mathcal{J}$ are comparable)

Assume that every chain $i_0 \le i_1 \le \cdots$ in $\mathcal{J}$ can be bounded above in $\mathcal{J}$,

ie $\exists\, j \in \mathcal{J}$ st $i_0 \le j,\ i_1 \le j,\ \dots,\ i_n \le j$ $\forall n \ge 0$.

Then, there exist maximal elements in $\mathcal{J}$.

In our case, • $\mathcal{J}$ = set of proper ideals of R which contain J.

$\mathcal{J} \neq \emptyset$ because $J \in \mathcal{J}$.

• $\le$ = inclusion ($I_1, I_2 \in \mathcal{J}$, $I_1 \le I_2$ means $I_1 \subseteq I_2$)

We verify the hypothesis of Zorn's Lemma: Assume we are given a chain in $\mathcal{J}$

$I_1 \subseteq I_2 \subseteq \cdots\cdots$ ie each $I_k \subsetneq R$ is a proper ideal containing J,

and $I_k \subseteq I_{k+1}$ $\forall k = 0, 1, 2, \dots$

Take $I = \sum_{k=0}^{\infty} I_k \underset{(*)}{=} \bigcup_{k=0}^{\infty} I_k$

(*) because $I_{k_1} + \cdots + I_{k_n} = I_{k_n}$ (since $I_{k_j} \subseteq I_{k_n}\ \forall j$)
$(k_1 < \cdots < k_n)$

To prove : (1) $I \subsetneq R$ is a proper ideal

(2) $I \supseteq J$ & $I \supseteq I_k$ $\forall k = 0, 1, 2, \dots$

(2) $I_k \supseteq J$ $\forall k$ so $J \subseteq \bigcup_{k=0}^{\infty} I_k = I$. Also, $I_k \subseteq I$ $\forall k$

(1) $I \neq R$ because if $1 \in I$, then $\exists k$ st $1 \in I_k$, contradicting the fact that I_k is proper.

Why is I an ideal?

$a, b \in I \Rightarrow \exists\, n \ge 0$ st $a, b \in I_n$ (hence, $a, b \in I_{n+\ell}$ $\forall \ell \ge 0$)

$$\left.\begin{array}{l} \Rightarrow a \pm b \in I_n \\ ra \in I_n \quad \forall r \in R \end{array}\right\} \Rightarrow \begin{array}{l} a \pm b \in I \\ ra \in I \quad \forall r \in R \end{array}$$

Hence, I is an ideal

Thus, Zorn's Lemma applies and we have maximal ideals $\square$

<u>Corollary</u>: Given a commutative ring R and a proper ideal J, $\exists\, I \subsetneq R$ prime ideal with $J \subseteq I$.

<u>Proof:</u> Use Proposition 1 and the fact that maximal ideals are prime (by Proposition (3) §40.3)

Proposition 2: (1) Any two distinct maximal ideals in a commutative ring R are coprime

(2) Let $f : R_1 \longrightarrow R_2$ be a ring homomorphism between two commutative rings R_1 & R_2. Let $P_2 \subsetneq R_2$ be a prime ideal. Then

$$P_1 = f^{-1}(P_2) = \{ a \in R_1 \mid f(a) \in P_2 \}$$

is again a prime ideal in R_1.

Proof: (1) If $M_1 \subsetneq R$ and $M_2 \subsetneq R$ are maximal ideals, then the ideal $M_1 + M_2$ contains both M_1 & M_2.

By maximality of M_1, we get $\overset{\overset{M_2}{\cap}}{M_1 + M_2} = M_1$ or $M_1 + M_2 = R$

Thus, $M_2 \subseteq M_1$ or $M_1 + M_2 = R$.

Since M_2 is maximal, M_1 is a proper ideal with $M_2 \neq M_1$, we cannot have $M_2 \subseteq M_1$. Therefore $M_1 + M_2 = R$ ie M_1 & M_2 are coprime.

(2) Consider the ring homomorphism

$$\begin{tikzcd} R_1 \arrow[r, "f"] \arrow[rr, bend right, "g"'] & R_2 \arrow[r, "\pi"] & R_2/P_2 \end{tikzcd}$$

$$a \longmapsto f(a) \bmod P_2 .$$

Since $g = \pi \circ f$ is a composition of ring homomorphisms, it is a ring hom. $\operatorname{Ker}(g) = \{ a \mid f(a) \in P_2 \} = f^{-1}(P_2)$, so by 1^{st} Isomorphism Theorem we get an injective ring homomorphism

$$R_1/P_1 \overset{i}{\hookrightarrow} R_2/P_2$$

indicates injective

But $P_2 \subsetneq R_2$ is prime $\Rightarrow$ R_2/P_2 is an integral domain.

Hence, the same is true for R_1/P_1 ($ab = 0$ in R_1/P_1 $\Rightarrow$ $i(a)\,i(b) = 0$ in R_2/P_2 $\Rightarrow$ $i(a) = 0$ or $i(b) = 0$ in R_2/P_2, ie $a = 0$ or $b = 0$ in R_1/P_1 because i is injective)

We conclude $P_1 \subsetneq R_1$ is prime. ($P_1 \neq R_1$ because $1_{R_1} \in R_1 \setminus P_1$, $f(1_{R_1}) = 1_{R_2} \notin P_2$) $\square$