Lecture XLII: Local rings

Recall the following result from §41.2 for commutative rings

Proposition: (1) Maximal ideals exist (Given $I \subsetneq R$ proper, $\exists\, M$ max'l ideal of R with $I \subseteq M$)

(2) $M_1, M_2 \subsetneq R$ maximal ideals $\Rightarrow$ ($M_1 = M_2$ or $M_1 + M_2 = R$)

(3) For any ring homomorphism $f: R_1 \longrightarrow R_2$ & $P_2 \subsetneq R_2$ prime ideal in R_2, the set $P_1 = f^{-1}(P_2) = \{a \in R_1 \mid f(a) \in P_2\} \subsetneq R_1$ is a prime ideal.

Remark: The conclusion of (3) will be false if we replace the adjective "prime" with "maximal"

Example:
$$\begin{array}{cccc} f: & \mathbb{Z} & \longrightarrow & \mathbb{Q} \\ & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ & n & \longmapsto & \frac{n}{1} \end{array}$$

$(0) \subsetneq \mathbb{Q}$ is (the only) maximal ideal

But $\underset{\{n : f(n) = 0\}}{\overset{\shortparallel}{(0)}} \subsetneq \mathbb{Z}$ is not a maximal ideal of $\mathbb{Z}$ (it's still prime though)

§42.1 Local Rings:

Let R be a commutative ring.

Definition: We say that R is <u>local</u> if it has a unique maximal ideal.
Usually we denote it by (R, M), if M is the unique maximal ideal of R.

Example: Every field is a local ring ($(0) \subsetneq K$ is the only proper ideal of K)

Proposition: A commutative ring R is local if, and only if

$$M = R \smallsetminus R^{\times} = \{a \in R \mid a \text{ is not a unit in } R\} \subsetneq R$$

is an ideal. In this case, M is the unique maximal ideal of R.

Proof: We begin by recalling that if $I \subsetneq R$ is a proper ideal, then $I \cap R^{\times} = \emptyset$ that is $I \subseteq M$. Thus, if $M \subsetneq R$ is an ideal, it is clearly the maximal idea.

($\Leftarrow$) Assume $M \subsetneq R$ is an ideal, then by the observation made above, we have
Set of Maximal ideals of $R = \{M\}$, ie R is local.

($\Rightarrow$) Assume R is a local ideal and let $J \subseteq R$ be its unique maximal ideal.
Then, $J \subseteq M$ ($J \cap R^{\times} = \emptyset$).

If $x \in M$, $I = (x) \subsetneq R$ because $I = \{rx \mid r \in R\}$ and $1 \in I \Leftrightarrow x \in R^{\times}$

Thus, $I \subsetneq R \Leftrightarrow x \in M = R \smallsetminus R^{\times}$.

By our previous result, $\exists\, \tilde{I} \subsetneq R$ maximal ideal containing I.

As R is local, $\tilde{I} = J$, hence $x \in J$ $\forall x \in M$, ie $M \subseteq J$.

<u>Conclusion:</u> $J \subseteq M$ & $M \subseteq J$ $\Rightarrow$ $M = J$ is an ideal of R □

<u>Example 1:</u> $R = K[x]/(x^2)$ is a local ring.

<u>Proof:</u> We determine $R^\times$. We can write any $f \in R$ as $f = a + bx$ with $a, b \in K$.

$a + bx \in R^\times$ with $a, b \in K$ $\iff$ $(a+bx)(c+dx) = 1$ for some $c, d \in K$, ie

$$1 = (a+bx)(c+dx) = ac + (ad+bc)x + \underbrace{bd\,x^2}_{=0} \iff \begin{cases} ac = 1 \\ ad + bc = 0 \end{cases}$$

We can solve these equations in K $\quad c = \frac{1}{a}$, $d = \frac{-bc}{a}$ $\iff a \in K^\times = K \setminus \{0\}$

Thus $R^\times = \{ a + bx \mid a \neq 0,\ a, b \in K \} \subseteq R$

$\Rightarrow R \setminus R^\times = (x)$

Since this is an ideal, we conclude that R is local & $(x) \subsetneq R$ is its unique maximal ideal.

Hence : Set of ideals of $R := K[x]/(x^2)$ $= \{ (0), (x), (1) = R \}$

only prime/maximal because $x^2 = 0$

<u>Alternative Argument:</u> Consider the natural projection $\pi : K[x] \twoheadrightarrow K[x]/(x^2)$

By the Second Isomorphism Theorem we have

$$\begin{array}{ccc} \left\{ \begin{array}{c} \text{Prime Ideals of } K[x] \\ \text{containing } (x^2) \end{array} \right\} & \longleftrightarrow & \left\{ \text{Prime Ideals of } R \right\} \\ \pi^{-1}(P) & \longleftarrow & P \\ Q & \longmapsto & \pi(Q) \end{array}$$

Furthermore, given $M \subseteq R$ maximal ideal, then M is prime, so $Q = \pi^{-1}(M)$ is a prime ideal of $K[x]$ containing x^2.

But $x^2 \in Q$ & Q is prime $\Rightarrow$ $x \in Q$ or $x \in Q$, ie $x \in Q$.

Thus $\{ Q \text{ prime ideal of } K[x] \text{ with } x^2 \in Q \} = \{ Q \text{ prime ideal of } K[x] \text{ with } x \in Q \}$

Now, $K[x]$ is a PID, so $Q = (f)$ for some $f \in K[x] \setminus K$ (because $K[x]^{\times} = K$).

Since $x \in Q = (f) \iff f \mid x$, we conclude that $\deg f \leq \deg x = 1$.

Thus, $f = ax$ for $a \in K$.

We conclude $M \subseteq R$ maximal $\Rightarrow M = \pi(\pi^{-1}(Q)) = \pi((x)) = (x) \subseteq R$.

Note: (x) is maximal because $R/(x) = \dfrac{K[x]/(x^2)}{(x)/(x^2)} \simeq \dfrac{K[x]}{(x)} \simeq K$ is a field.

Thus, R is a local ring because (x) is its unique maximal ideal.

Example 2: $R = K[[x]]$ ring of formal power series in one variable with coefficients from a field K

A typical element of R is of the form:

$$a_0 + a_1 x + a_2 x^2 + \cdots = \sum_{n=0}^{\infty} a_n x^n$$

Addition: componentwise $\left(\sum_{n=0}^{\infty} a_n x^n\right) + \left(\sum_{k=0}^{\infty} b_k x^k\right) = \sum_{n=0}^{\infty} (a_n + b_n) x^n$

Multiplication: distributive over +

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{k=0}^{\infty} b_k x^k\right) = \sum_{N=0}^{\infty} \left(\sum_{n+k=N} a_n b_k\right) x^N = \sum_{N=0}^{\infty} \left(\sum_{j=0}^{N} a_j b_{N-j}\right) x^N.$$

Note: In our definition of + and $\cdot$ of elements of $R = K[[x]]$ only finitely many operations where performed to get the coefficients of x^n (for a fixed n)

E.g. $\sum_{j=0}^{N} a_j b_{N-j} \in K$ for $\begin{matrix} a_0, \ldots, a_N \\ b_0, \ldots, b_N \end{matrix} \in K$

The same idea will give us a ring structure on the ring of Laurent series

$$K((X)) := \left\{ \sum_{j=-N}^{\infty} a_j x^j \;:\; a_j \in K \quad j = -N, -N+1, \ldots, N \in \mathbb{Z}_{\geq 0} \right\}.$$

Alternative notation: $K[x^{-1}, x]]$

Fun exercise: Our definition of multiplication will not make sense for the abelian group $K[[x^{-1}; x]] := \left\{ \sum_{j=-\infty}^{\infty} a_j x^j \;:\; a_j \in K \;\; \forall j \in \mathbb{Z} \right\}$ (group under +)

<u>Reason</u>: if it did $\cdots + x^{-2} + x^{-1} + 1 + x + \cdots = \frac{x^{-1}}{1-x^{-1}} + \frac{1}{1-x} = 0$

which can't be true (compare coefficient of x^k to get $1=0$).

<u>Claim</u>: $R = K[\![x]\!]$, $R^\times = K^\times + xK[\![x]\!]$ ie power series with non-zero constant term. Reason: we can compute f^{-1} by hand term by term if $f(0)\neq 0$

$\Rightarrow R \smallsetminus R^\times = (x)$ is an ideal, hence R is a local ring.

<u>Example 3</u>: Fix $p \in \mathbb{Z}_{\geq 2}$ a prime & define

$$R = \left\{ \tfrac{a}{b} \in \mathbb{Q} \;\middle|\; \begin{array}{l} \gcd(a,b)=1 \\ p \text{ does not divide } b \end{array} \right\} \qquad (\text{Notation: } \mathbb{Z}_{(p)})$$

<u>Claim 1</u>: R is a subring of $\mathbb{Q}$

Pf/ $0 = \frac{0}{1}$, $1 = \frac{1}{1} \in R$

- $\frac{a_1}{b_1}, \frac{a_2}{b_2} \in R \Rightarrow \frac{a_1}{b_1} \pm \frac{a_2}{b_2} = \frac{a_1b_2 \pm a_2 b_1}{b_1 b_2}$ & $\frac{a_1}{b_1} \cdot \frac{a_2}{b_2} = \frac{a_1 a_2}{b_1 b_2}$

  $p \nmid b_1,\ p\nmid b_2 \Rightarrow p \nmid b_1 b_2$ so after simplifying $\frac{a_1}{b_1} \pm \frac{a_2}{b_2}$ & $\frac{a_1}{b_1}\frac{a_2}{b_2}$ in $\mathbb{Q}$

We conclude both rational numbers lie in R.

<u>Claim 2</u>: $R^\times = \left\{ \frac{a}{b} \in \mathbb{Q} \;\middle|\; \begin{array}{l} \gcd(a,b)=1 \\ p\nmid a \ \& \ p \nmid b \end{array} \right\} \subseteq R$ $\left(\left(\frac{a}{b}\right)^{-1} = \frac{b}{a} \text{ in } \mathbb{Q} \text{ \& lies in } R \right)$

$\Rightarrow R \smallsetminus R^\times = (p) = pR$ is an ideal

<u>Conclude</u>: R is a local ring.

<u>Summary</u>: The 3 examples above show how to obtain local rings:

$K[x]/(x^2)$	$K[\![x]\!]$	$\mathbb{Z}_{(p)}$
Quotient by an ideal contained in a unique maximal ideal.	Completion with respect to the ideal (x)	Localization away from (p) (special case of forming ring of fractions)