Lecture XLIII: Ring of Fractions

§43.1 Ring of Fractions:

Let R be a commutative ring and $S \subseteq R$.

Definition: We say S is multiplicatively closed if

(1) $0 \notin S$

(2) $1 \in S$

(3) $a, b \in S \implies a \cdot b \in S$.

We are going to use S to define a new ring, denoted by $S^{-1}R$ by "inverting elements of S"

- Model to keep in mind: how $\mathbb{Q}$ is defined starting from $\mathbb{Z}$ and inverting every non-zero $n \in \mathbb{Z}_{\neq 0}$. Here, $S = \mathbb{Z} \setminus \{0\}$.

Definition: $S^{-1}R = (S \times R)/\sim$ where $\sim$ is an equivalence relation on $S \times R$ defined as follows.

Given $(s_1, r_1) \in S \times R$, $(s_2, r_2) \in S \times R$, we say $(s_1, r_1) \sim (s_2, r_2)$ if there exists $t \in S$ such that $\boxed{t(s_1 r_2 - r_1 s_2) = 0}$ $(*)$

⚠ In the textbook it is further assumed that a multiplicatively closed set does NOT contain zero-divisors. With this stronger assumption, we don't need to include 't' in the definition of $\sim$.

We will not be making this stronger assumption (in particular, S may contain a zero-divisor of R).

Lemma: The relation $\sim$ defined above is an equivalence relation.

Proof: We need to check $\sim$ is reflexive, symmetric and transitive.

(i) $(s,r) \sim (s,r)$ $\forall s \in S$ $\forall r \in R$ $(t = 1 \in S$ & $1 \cdot (sr - rs) = 0)$

(ii) $(s_1,r_1)\sim(s_2,r_2)$ $\quad \exists\, t\in S: t(s_1r_2-r_1s_2) = (-t)(s_2r_1-r_2s_1)=0$

$\Rightarrow t(s_2r_1-r_2s_1)=0$ & $t\in S$, so $(s_2,r_2)\sim(s_1,r_1)$

(iii) $(s_1,r_1)\sim(s_2,r_2)$ & $(s_2,r_2)\sim(s_3,r_3)$

$\Downarrow$ $\exists\, t\in S: t(s_1r_2-s_2r_1)=0$ $\quad$ $\Downarrow$ $\exists\, t'\in S: t'(s_2r_3-r_2s_3)=0$

We need to find $t''\in S$ with $t''(s_1r_3-r_1s_3)=0$

Pick $t''=tt's_2 \in S$ (because $t,t',s_2\in S$ & S is multiplicatively closed)

$$\Rightarrow t''(s_1r_3-s_3r_1) = tt's_2(s_1r_3-s_3r_1) = t\underbrace{t's_2r_3}_{t'r_2s_3}s_1 - t'\underbrace{ts_2r_1}_{ts_1r_2}s_3$$

$$= tt'r_2s_3s_1 - t'ts_1r_2s_3 = 0.$$

Since $t''\in S$ & $t''(s_1r_3-s_3r_1)=0$, then $(s_1,r_1)\sim(s_3,r_3)$.

Notation: A typical element of $S^{-1}R$ is an equivalence class in $S\times R$ modulo $\sim$

$\frac{r}{s} = [(s,r)]$ equivalence class modulo $\sim$ ie $=\{(s',r')\in S\times R \mid (s',r')\sim(s,r)\}$

In particular:

$$\frac{r}{s}=\frac{0}{1} \iff \exists\, t\in S \text{ with } t(s\cdot 0-1\cdot r) = -tr=0 \iff \exists\, t\in S: t\cdot r=0$$

§43.2 Ring structure on $S^{-1}R$:

We define addition and multiplication on $S^{-1}R$ modeled on $\mathbb{Q}$.

- $\frac{r_1}{s_1}+\frac{r_2}{s_2} = \frac{s_2r_1+s_1r_2}{s_1s_2}$
- $\frac{r_1}{s_1}\cdot\frac{r_2}{s_2} = \frac{r_1r_2}{s_1s_2}$
- Neutral elements $0_{S^{-1}R}=\frac{0}{1}$ & $1_{S^{-1}R}=\frac{1}{1}$

We need to check these operations are well-defined:

Proposition: (1) The operations on $S^{-1}R$ are well-defined

(2) $(S^{-1}R, +, \frac{0}{1})$ is an abelian group

(3) $\cdot$ is associative & commutative, $\frac{1}{1}$ is multiplicatively neutral, $\frac{1}{1}\neq\frac{0}{1}$

(4) $\cdot$ distributes over addition.

Proof: (1) $(s_1, r_1) \sim (s_1', r_1')$ & $(s_2, r_2) \sim (s_2', r_2')$

We know $\exists\, t \in S$ $\quad t(s_1 r_1' - r_1 s_1') = 0$

$\exists\, t' \in S$ $\quad t'(s_2 r_2' - r_2 s_2') = 0$

$+$ Then, $\dfrac{r_1}{s_1} + \dfrac{r_2}{s_2} = \dfrac{r_1 s_2 + s_1 r_2}{s_1 s_2} \overset{?}{=} \dfrac{r_1' s_2' + s_1' r_2'}{s_1' s_2'} = \dfrac{r_1'}{s_1'} + \dfrac{r_2'}{s_2'}$

We want to show: $(s_1 s_2, r_1 s_2 + s_1 r_2) \sim (s_1' s_2', r_1' s_2' + s_1' r_2')$, ie $\exists\, t'' \in S$

with $t''(s_1 s_2 (r_1' s_2' + s_1' r_2') - s_1' s_2' (r_1 s_2 + s_1 r_2)) = 0$

$t''(s_1 s_2 r_1' s_2' + s_1 s_1' s_2 r_2' - s_1' s_2' r_1 s_2 - s_1' s_1 r_2 s_2') = 0$

$t''(s_2 s_2' (s_1 r_1' - s_1' r_1) + s_1 s_1' (s_2 r_2' - s_2' r_2)) = 0$

Pick $t'' = tt' \in S$ (S is multiplicatively closed)

$\Rightarrow t''(s_1 s_2 (r_1' s_2' + s_1' r_2') - s_1' s_2' (r_1 s_2 + s_1 r_2)) =$

$= tt'(s_2 s_2' (s_1 r_1' - s_1' r_1) + s_1 s_1' (s_2 r_2' - s_2' r_2)) =$

$= s_2 s_2' t' \underbrace{t(s_1 r_1' - s_1' r_1)}_{=0} + s_1 s_1' t' \underbrace{t(s_2 r_2' - s_2' r_2)}_{=0} = 0$

$\cdot$ Then, $\dfrac{r_1}{s_1} \cdot \dfrac{r_2}{s_2} = \dfrac{r_1 r_2}{s_1 s_2} \overset{?}{=} \dfrac{r_1' r_2'}{s_1' s_2'} = \dfrac{r_1'}{s_1'} \cdot \dfrac{r_2'}{s_2'}$

We want to show: $(s_1 s_2, r_1 r_2) \sim (s_1' s_2', r_1' r_2')$ ie $\exists\, t'' \in S$ with

$t''(s_1 s_2 r_1' r_2' - r_1 r_2 s_1' s_2') = 0$

Pick $t'' = tt' \in S$ (S is multiplicatively closed)

$\Rightarrow t''(s_1 s_2 r_1' r_2' - r_1 r_2 s_1' s_2') = tt'(s_1 s_2 r_1' r_2' - r_1 r_2 s_1' s_2') =$

$= \underbrace{t s_1 r_1'}_{= t r_1 s_1'} t' s_2 r_2' - t r_1 s_1' \underbrace{t' r_2 s_2'}_{= t' s_2 r_2'} = t' t r_1 s_1' s_2 r_2' - t' t r_1 s_1' s_2 r_2' = 0$ $\square$

(2) The formula for $+$ is symmetric so $+$ is commutative.

Also, $+$ is associative (inherited by associativity of $+$ and $\cdot$ in R)

$\dfrac{0}{1}$ is neutral : $\dfrac{0}{1} + \dfrac{r}{s} = \dfrac{0 \cdot s + r \cdot 1}{1 \cdot s} = \dfrac{r \cdot 1}{1 \cdot s} = \dfrac{r}{s}$ ✓

$-\dfrac{r}{s} = \dfrac{-r}{s}$ since $\dfrac{r}{s} + \dfrac{-r}{s} = \dfrac{r \cdot s + (-r)s}{s^2} = \dfrac{0}{s^2} = \dfrac{0}{1}$ because

$1(1 \cdot 0 - 0 \cdot s^2) = 1 \cdot 0 = 0$ & $1 \in S$.

(3) The formula for $\cdot$ is symmetric so $\cdot$ is commutative.

Also, $\cdot$ is associative (inherited by associativity of $\cdot$ in R)

$\frac{1}{1}$ is Neutral : $\frac{1}{1}\cdot\frac{r}{s} = \frac{1\cdot r}{1\cdot s} = \frac{r}{s}$

$\frac{1}{1} \neq \frac{0}{1} \iff \nexists\, t\in S \quad 0 = t(1\cdot 1 - 0\cdot 1) = t\cdot 1 = t$

This is so because $0\notin S$.

(4) Multiplication distributes over addition

$$\frac{r}{s}\left(\frac{r_1}{s_1}+\frac{r_2}{s_2}\right) = \frac{r}{s}\left(\frac{r_1s_2+r_2s_1}{s_1s_2}\right) = \frac{r(r_1s_2+r_2s_1)}{ss_1s_2} \overset{?}{=} \frac{rr_1s_2+rr_2s_1}{ss_1s_2}$$

$$= \frac{rr_1ss_2+rr_2ss_1}{s^2s_1s_2} = \frac{r}{s}\frac{r_1}{s_1}+\frac{r}{s}\frac{r_2}{s_2}$$

$\iff (ss_1s_2,\ rr_1s_2+rr_2s_1) \sim (s^2s_1s_2,\ rr_1ss_2+rr_2ss_1)$

$\iff \exists\, t\in S$ with $t\left(ss_1s_2(rr_1ss_2+rr_2ss_1) - (rr_1s_2+rr_2s_1)s^2s_1s_2\right)$

$= t\left(ss_1s_2rr_1ss_2 + ss_1s_2rr_2ss_1 - s^2s_1s_2rr_1s_2 - s^2s_1s_2rr_2s_1\right)$

$= ts^2\left(s_2^2rr_1s_1 + s_1^2rs_2r_2 - s_2^2rr_1s_1 - s_1^2rs_2r_2\right) = ts^2(0-0) = 0$

Consequence :

INPUT

R : commutative ring

$S\subset R$ mult. closed

$[0\notin S;\ 1\in S;\ a,b\in S \Downarrow ab\in S]$

$\rightsquigarrow$ "invert S"

OUTPUT

$S^{-1}R := \left\{\frac{r}{s} : r\in R, s\in S\right\} \Big/ \left(\frac{r}{s}=0\right)$

$\Updownarrow$

$\exists t\in S$ with $t\cdot r = 0$

(ie adding $\frac{r}{s}=0$ to any $\frac{r'}{s'}$ will give the same element in $S^{-1}R$.)

Name: $S^{-1}R$ ring of fractions

§43.3 Examples:

(1) $R=\mathbb{Z}$ (or any integral domain)

$S = R\setminus\{0\}$

$\rightsquigarrow S^{-1}R = (R\setminus\{0\})R = \mathbb{Q}$ $\quad\left(\frac{r}{s}=\frac{r'}{s'} \iff \underset{\neq 0}{t}(sr'-s'r)=0 \quad\right)$ $\Rightarrow sr'-s'r=0$

R is an integral domain $\iff [ab=0 \Rightarrow a=0 \text{ or } b=0]$

ie $(0)\subsetneq R$ is a prime ideal , ie $R\setminus\{0\} = S\subseteq R$ is multiplicatively closed

Notation: R : integral domain

$F(R)$ = "field of fractions of R" (F for Fractions)

$= (R\setminus\{0\})^{-1} R$

Lemma 1: $F(R)$ is a field

Proof: It is a field because $x = \frac{r}{s} \in F(R) \Rightarrow$ either $r=0$ ie $x=0$ or $y = \frac{s}{r} \in F(R)$ is x^{-1}.

In particular, $F(R)^{\times} = F(R)\setminus\{0\}$. $\square$

② A non-integral example:

$R = \mathbb{Z}/6\mathbb{Z} \supset S = \{1,2,4\} = \{2^n : n \in \mathbb{Z}_{\geq 0}\}$

In $S^{-1}R$, $\frac{r}{s} = 0 \iff \exists\, t \in S$ with $tr = 0$

So $\frac{3}{s} = \frac{0}{s'}$ $\forall s, s' \in S$. Meaning, $\exists$ 6 elements in each $\sim$ in $S \times R$: $\frac{0}{s}, \frac{3}{s}$ $(s = 1,2,4)$

There are $\frac{18}{6} = 3$ elements in $S^{-1}R$ (representatives $\frac{0}{1}, \frac{1}{1}, \frac{1}{2}$)

Indeed:

- $\frac{0}{1}+\frac{3}{1} = \frac{3}{1}$, $\frac{0}{1}+\frac{3}{2} = 3/2$, $\frac{0}{1}+\frac{3}{4} = 3/4$; $\frac{0}{1}+\frac{0}{1} = 0/1$; $\frac{0}{1}+\frac{0}{2} = 0/2$, $\frac{0}{1}+\frac{0}{4} = 0/4$ $\quad\left[\frac{0}{1}\right]$
- $\frac{1}{1}+\frac{3}{1} = \frac{1+3}{1} = \frac{4}{1}$, $\frac{1}{1}+\frac{3}{2} = \frac{5}{1}$, $\frac{1}{1}+\frac{3}{4} = \frac{1}{4}$; $\frac{1}{1}+\frac{0}{1} = \frac{1}{1}$, $\frac{1}{1}+\frac{0}{2} = \frac{2}{2}$, $\frac{1}{1}+\frac{0}{4} = \frac{4}{4}$ $\quad\left[\frac{1}{1}\right]$
- $\frac{1}{2}+\frac{3}{1} = \frac{5}{2}$, $\frac{1}{2}+\frac{3}{2} = \frac{2}{4}$, $\frac{1}{2}+\frac{3}{4} = \frac{4}{2}$; $\frac{1}{2}+\frac{0}{1} = \frac{1}{2}$; $\frac{1}{2}+\frac{0}{2} = \frac{2}{4}$; $\frac{1}{2}+\frac{0}{4} = \frac{4}{2} = \frac{2}{1}$ $\quad\left[\frac{1}{2}\right]$

Check which elements we obtained:

$$\underbrace{\tfrac{0}{1}, \tfrac{0}{2}, \tfrac{0}{4}}_{\checkmark} \quad \underset{\checkmark}{\tfrac{1}{1}}, \underset{\checkmark}{\tfrac{1}{2}}, \underset{\checkmark}{\tfrac{1}{4}}, \underset{\checkmark}{\tfrac{2}{1}}, \underset{\checkmark}{\tfrac{2}{2}}, \underset{\checkmark}{\tfrac{2}{4}}, \underbrace{\tfrac{3}{1}, \tfrac{3}{2}, \tfrac{3}{4}}_{\checkmark}, \underset{\checkmark}{\tfrac{4}{1}}, \underset{\checkmark}{\tfrac{4}{2}}, \underset{\checkmark}{\tfrac{4}{4}}, \underset{\checkmark}{\tfrac{5}{1}}, \underset{\checkmark}{\tfrac{5}{2}}, \tfrac{5}{4} = 1/2$$

$\frac{0}{1}+\frac{0}{1} = \frac{0}{1}$; $\frac{0}{1}+\frac{1}{1} = \frac{1}{1}$; $\frac{0}{1}+\frac{1}{2} = \frac{1}{2}$; $-\frac{1}{1} = \frac{2}{1}$

$\frac{1}{1}+\frac{0}{1} = \frac{1}{1}$; $\frac{1}{1}+\frac{1}{1} = \frac{2}{1}$; $\frac{1}{1}+\frac{1}{2} = \frac{3}{2} = \frac{0}{1}$; $-\frac{1}{2} = \frac{2}{2} = \frac{1}{1}$

$\frac{1}{2}+\frac{0}{1} = \frac{1}{2}$; $\frac{1}{2}+\frac{1}{1} = \frac{3}{2} = \frac{0}{1}$; $\frac{1}{2}+\frac{1}{2} = \frac{4}{4} = \frac{1}{1}$; $-\frac{0}{1} = \frac{0}{1}$

$\frac{0}{1}\cdot\frac{0}{1} = \frac{1}{1}\cdot\frac{0}{1} = \frac{1}{2}\cdot\frac{0}{1} = \frac{0}{1}$; $\frac{1}{1}\cdot\frac{1}{1} = \frac{1}{1}$; $\frac{1}{1}\cdot\frac{1}{2} = \frac{1}{2}$; $\frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4} = \frac{1}{1}$