Lecture XLIV: Properties of Localization

Recall: We constructed a ring of fractions

$S \subsetneq R$ mult. closed set in a commutative ring $\rightsquigarrow$ $S^{-1}R = \left\{ \frac{r}{s} : \begin{matrix} r \in R \\ s \in S \end{matrix} \right\} \Big/ \left(\frac{r}{s} = 0 \Leftrightarrow tr = 0 \text{ for some } t \in S. \right)$

Special cases:

① R integral domain $\Rightarrow$ we don't need t.

$R \supseteq R \smallsetminus \{0\}$ mult. closed set $\rightsquigarrow$ $F(R) = (R \smallsetminus \{0\})^{-1} R$ field of fractions of R

② $R = \mathbb{Z}/6\mathbb{Z}$ $\quad S = \{1, 2, 4\}$ $\rightsquigarrow$ $S^{-1}R = \{ \underset{\overset{\shortparallel}{0}}{\frac{0}{1}}, \underset{\overset{\shortparallel}{1}}{\frac{1}{1}}, \frac{1}{2} \}$

$\begin{cases} \frac{1}{2} + \frac{1}{2} = \frac{1}{1} & \frac{1}{1} + \frac{1}{1} = \frac{1}{2} \\ \frac{1}{1} + \frac{1}{2} = \frac{0}{1} & \\ \frac{1}{2} \cdot \frac{0}{1} = \frac{0}{1} & \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{1} \end{cases}$

③ $R = \mathbb{Z} \supsetneq P = p\mathbb{Z}$ where $p \in \mathbb{Z}_{\geq 2}$ is prime

$\mathbb{Z} \smallsetminus p\mathbb{Z} = \{ n \in \mathbb{Z} : \gcd(n,p) = 1 \}$ $\qquad \gcd(n,p) = 1$ or $p \nmid n$ or $p \notin n\mathbb{Z}$

$\rightsquigarrow \mathbb{Z}_{(p)} := (\mathbb{Z} \smallsetminus p\mathbb{Z})^{-1} \mathbb{Z} = \left\{ \frac{a}{b} \in \mathbb{Q} \mid \begin{matrix} \gcd(a,b) = 1 \\ p \nmid b \end{matrix} \right\}$

This will generalize to a pair R = any commutative ring $\cup\!\!+$ $\mathfrak{P}$ = any prime ideal

Lemma: $S = R \smallsetminus \mathfrak{P}$ is multiplicatively closed

Proof:
- $a, b \in S \Leftrightarrow a, b \notin \mathfrak{P} \underset{\mathfrak{P} \text{ prime}}{\Rightarrow} ab \notin \mathfrak{P} \Leftrightarrow ab \in S$
- $0 \in \mathfrak{P} \Rightarrow 0 \notin S$
- $1 \notin \mathfrak{P} \Rightarrow 1 \in S$

$\square$

We use the notation $R_{\mathfrak{P}} = (R \smallsetminus \mathfrak{P})^{-1} R$ "R localized at $\mathfrak{P}$"

Elements of $R_{\mathfrak{P}}$: $\frac{r}{s}$ where $r \in R$ & $s \notin \mathfrak{P}$

relation $\frac{r}{s} = 0 \Leftrightarrow t \cdot r = 0$ for some $t \notin \mathfrak{P}$. $(\Rightarrow r \in \mathfrak{P})$

Again, for $x = \frac{r}{s}$ we have two cases: $r \in \mathfrak{P}$ or $r \notin \mathfrak{P}$

$\Downarrow$

$\frac{s}{r} = y \in R_{\mathfrak{P}}$ is x^{-1}.

Conclusion: $R_{\mathfrak{p}}^{\times} = \{ \frac{r}{s} : r \notin \mathfrak{p}, s \notin \mathfrak{p} \} \Longrightarrow R_{\mathfrak{p}} \setminus R_{\mathfrak{p}}^{\times} = \{ \frac{r}{s} : r \in \mathfrak{p}, s \notin \mathfrak{p} \}$
$= \text{ideal in } R_{\mathfrak{p}} \text{ generated by } \frac{\mathfrak{p}}{1}$.

Then $R_{\mathfrak{p}} \setminus R_{\mathfrak{p}}^{\times} = \mathfrak{p} R_{\mathfrak{p}}$ is an ideal in $R_{\mathfrak{p}}$.
Hence $(R_{\mathfrak{p}}, \mathfrak{p} R_{\mathfrak{p}})$ is a local ring, called the localization of R at $\mathfrak{p}$.

Summary: R commutative ring $\supseteq S = R \setminus \mathfrak{p}$: mult. closed set
$\cup\!|$
$\mathfrak{p}$: prime ideal

$\rightsquigarrow (R_{\mathfrak{p}}, \mathfrak{p} R_{\mathfrak{p}})$ is a local ring called localization of R at $\mathfrak{p}$.

§44.1 More properties of $S^{-1}R$:

Proposition 1: We have a ring homomorphism $j: R \longrightarrow S^{-1}R$, $r \longmapsto \frac{r}{1}$

Moreover
- $\operatorname{Ker} j := \{ r \mid t \cdot r = 0 \text{ for some } t \in S \}$
- $j(s) = \frac{s}{1}$ is invertible in $S^{-1}R$ $\forall s \in S$.

Proof:
- By our definition of $+$ & $\cdot$ in $S^{-1}R$, j respects $+$ & $\cdot$. Also, $j(1) = \frac{1}{1} = \mathbb{1} \in S^{-1}R$
- $\frac{r}{1} = \frac{0}{1} \iff \exists t \in S : t(1 \cdot 0 - r \cdot 1) = -tr = 0 \iff tr = 0$.
- $j(s) = \frac{s}{1}$ is invertible for $s \in S$, because $\frac{1}{s} \in S^{-1}R$ and $\frac{s}{1} \cdot \frac{1}{s} = \frac{s}{s} = \frac{1}{1}$.

($t = 1$ gives $1 \cdot (s \cdot 1 - s \cdot 1) = 0$ ✓) □

⚠ We should NOT expect j to be injective, unless R is an integral domain or S has no zero-divisors.

Example: $R = \mathbb{Z}/6\mathbb{Z}$ $S = \{1, 2, 4\}$ $j(3) = \frac{3}{1} = \frac{0}{1}$ in $S^{-1}R$
Furthermore $|R| = 6$, $|S^{-1}R| = 3$, so it can't be injective.

Remark: The properties in Proposition 1 are the "defining properties" of $S^{-1}R$

Meaning: 1^{st} Iso Theorem, in a very precise sense, defined the quotient R/I. It says that R/I is the only ring which comes together with a surjective ring$_{(1)}$ homomorphism $\pi: R \to R/I$

and satisfies: "every ring homomorphism $f: R \longrightarrow R'$ such that $I \subset \operatorname{Ker} f_{(2)}$

factorizes (uniquely!) through
$$\begin{tikzcd} R \arrow[r, "f"] \arrow[dr, "\pi"'] & R' \\ & R/I \arrow[u, "\exists! F"'] \end{tikzcd}$$
"

The following result is the analog of the 1^{st} Iso Theorem. For the ring of fractions $S^{-1}R$ replace (1) by j st $j(S) \subset (S^{-1}R)^\times$

(2) by $f(S) \subseteq (R')^\times$.

Here the precise statement:

Proposition 2: Let R be a commutative ring and S a multiplicatively closed subset. Let $f: R \longrightarrow R'$ be a ring homomorphism such that $f(S) \subset (R')^\times$. Then, $\exists!$ ring homomorphism denoted by $\tilde{f}$ below:
$$\begin{aligned} S^{-1}R &\xrightarrow{\tilde{f}} R' \\ \tfrac{r}{s} &\longmapsto f(s)^{-1} f(r) \end{aligned}$$

such that
$$\begin{tikzcd} R \arrow[rr, "f"] \arrow[dr, "j"'] & \arrow[d, phantom, "\circlearrowleft"] & R' \\ & S^{-1}R \arrow[ur, "\exists! \tilde{f}"'] & \end{tikzcd}$$

Proof: $\tilde{f}(\frac{r}{1}) = f(1)^{-1} f(r) = 1^{-1} f(r) = f(r)$ so $f = \tilde{f} \circ j$.

- $\tilde{f}$ is a ring homomorphism:

$$\tilde{f}\left(\frac{r}{s} + \frac{r'}{s'}\right) = \tilde{f}\left(\frac{rs' + sr'}{ss'}\right) = f(ss')^{-1} f(rs' + sr')$$
$$= f(s)^{-1} f(s')^{-1} \left(f(r) f(s') + f(s) f(r') \right)$$
$$= f(s)^{-1} f(r) + f(s')^{-1} f(r') = \tilde{f}\left(\frac{r}{s}\right) + \tilde{f}\left(\frac{r'}{s'}\right).$$
$$\tilde{f}\left(\frac{r}{s} \cdot \frac{r'}{s'}\right) = \tilde{f}\left(\frac{rr'}{ss'}\right) = f(ss')^{-1} f(rr') = f(s)^{-1} f(s')^{-1} f(r) f(r')$$
$$= f(s)^{-1} f(r) \, f(s')^{-1} f(r') = f\left(\frac{r}{s}\right) f\left(\frac{r'}{s'}\right).$$
$$\tilde{f}\left(\frac{1}{1}\right) = f(1)^{-1} f(1) = 1^{-1} \cdot 1 = 1 \in R'.$$

- If $g: S^{-1}R \longrightarrow R'$ ring homomorphism with $f = g \circ j$, then
$$g\left(\frac{r}{s}\right) = g\left(\frac{r}{1}\right) g\left(\frac{1}{s}\right) = g\left(\frac{r}{1}\right) g(s^{-1}) = g\left(\frac{r}{1}\right) g\left(\frac{s}{1}\right)^{-1} = g\left(\frac{r}{1}\right) g\left(\frac{s}{1}\right)^{-1}$$
$$= f(r) f(s)^{-1} = \tilde{f}\left(\frac{r}{s}\right)$$

Thus, $\tilde{f}$ is unique. $\square$

§ 44.2 Ideals of $S^{-1}R$:

We assume R is a commutative ring & $S \subseteq R$ is a multiplicatively closed set.

We consider the ring homomorphism $j: R \longrightarrow S^{-1}R$

Definition: Given an ideal $I \subseteq R$ we define:

$$S^{-1}I = (j(I))_{S^{-1}R} = \text{ideal of } S^{-1}R \text{ defined by } j(I).$$

↳ ideal in $S^{-1}R$ generated by $\{\frac{a}{1} : a \in I\}$

Lemma 1: Let $I \subseteq R$ be an ideal.

(1) $S^{-1}I = \{\frac{a}{s} : a \in I, s \in S\}$

(2) Every ideal in $S^{-1}R$ is of the form $S^{-1}I$ for some ideal $I \subseteq R$.

Proof: Recall $Ker(j) = \{r \in R \mid \exists\, t \in S \text{ with } tr = 0\} \subsetneq R$ is an ideal of R

Notation: For an ideal $\tilde{I} \subseteq S^{-1}R$ we define

$$j^*(\tilde{I}) := j^{-1}(\tilde{I}) = \{a \in R \mid j(a) = \tfrac{a}{1} \in \tilde{I}\} \subseteq R$$

We know by Lemma 1 § 38.2 that $j^*(\tilde{I})$ is an ideal of R.

(1) We define $J = \{\frac{a}{s} : a \in I \ s \in S\}$

We show $J \subseteq S^{-1}R$ is an ideal:

- $\frac{0}{1} \in J$ because $0 \in I$, $1 \in S$
- $x_1 = \frac{a_1}{s_1}, x_2 = \frac{a_2}{s_2} \in J$ $(a_1, a_2 \in I, \ s_1, s_2 \in S) \Rightarrow x_1 \pm x_2 = \dfrac{s_2 a_1 \pm a_2 s_1}{s_1 s_2}$

but $s_2 \underset{I}{\underset{\cap}{a_1}} \pm \underset{I}{\underset{\cap}{a_2}} s_1 \in I$ & $s_1 \cdot s_2 \in S$ so $x_1 \pm x_2 \in J$.

- $x_1 = \frac{a_1}{s_1} \in J$ $\left(\begin{matrix} a_1 \in I \\ s_1 \in S \end{matrix}\right)$ and $\frac{r}{s} \in S^{-1}R \Rightarrow \frac{r}{s} \cdot \frac{a_1}{s_1} = \frac{ra_1}{ss_1}$

Since $\begin{matrix} a_1 \in I \\ s_1, s \in S \end{matrix}$, then $\begin{matrix} ra_1 \in I \\ s_1 s \in S \end{matrix}$ **Conclusion:** $\frac{r}{s} x_1 = \frac{ra_1}{ss_1} \in J$

Hence, J is an ideal of $S^{-1}R$.

- Next, we show $(j(I))_{S^{-1}R} = J$

(⊆) $j(I) \subseteq J$ because $a \in I$ & $s = 1 \in S$. Since J is an ideal, we get $(j(I))_{S^{-1}R} \subseteq J$ by definition of ideal generated by a set

(⊇) Given $x = \frac{a}{s} \in J$ with $a \in I$, $s \in S$ then

$$x = \frac{a}{s} = \underbrace{\frac{1}{s}}_{\in S^{-1}R} \cdot \underbrace{\boxed{\frac{a}{1}}}_{\in j(I)} \in (j(I))_{S^{-1}R}.$$

Thus $J \subseteq (j(I))_{S^{-1}R}$.

(2) Given $\tilde{I} \subseteq S^{-1}R$ an ideal, we let $I = j^*(\tilde{I}) \subseteq R$.

By construction, I is an ideal of R. Furthermore, $I = \{a \in R \mid \frac{a}{1} \in \tilde{I}\}$

<u>Claim</u>: $S^{-1}I = \tilde{I}$:

Pf/ We check the double inclusion:

(⊆) $S^{-1}I \underset{\text{part (1)}}{=} (j(I))_{S^{-1}R} \subseteq (\tilde{I})_{S^{-1}R} \underset{\tilde{I}\text{ is an ideal}}{=} \tilde{I}$

$$\begin{array}{ccc} R & \xrightarrow{j} & S^{-1}R \\ \cup\!\!\mid & & \cup\!\!\mid \\ I = j^*\tilde{I} & \longrightarrow & \tilde{I} \end{array}$$

(⊇) Conversely, given $\frac{r}{s} \in \tilde{I}$, we have $\frac{s}{1} \cdot \frac{r}{s} \in \tilde{I}$, ie $\frac{r}{1} \in \tilde{I}$ $\left(\frac{sr}{s} = \frac{r}{1}\right)$

Hence $j(r) = \frac{r}{1} \in \tilde{I} \implies r \in j^*(\tilde{I})$, so $\frac{r}{s} \in S^{-1}I$.

Conclude $\tilde{I} \subseteq S^{-1}I$