Lecture XLV: Localizations and their (prime) ideals

§45.1 Summary:

Recall: Our recent construction

R : commutative ring
$\cup$
S : multiplicatively closed set
($1 \in S$; $0 \notin S$;
$a, b \in S \Rightarrow ab \in S$)

$\rightsquigarrow$

$S^{-1}R$: ring of fractions
&
$$j : R \longrightarrow S^{-1}R$$
$$r \longmapsto \frac{r}{1}$$
$$\operatorname{Ker} j = \{a \in R : \exists t \in S \text{ with } ta = 0\}$$

We prove the following facts:

Proposition 1: $\forall\, I \subseteq R$ ideal, the set $S^{-1}(I) := \{\frac{a}{s} : a \in I, s \in S\} \subseteq S^{-1}R$ is an ideal. Furthermore, $S^{-1}(I)$ = ideal of $S^{-1}R$ generated by $j(I) = \{j(a) : a \in I\}$

Proposition 2: Every ideal $\tilde{I} \subseteq S^{-1}R$ is of the form $S^{-1}(I)$ for some ideal $I \subseteq R$. More precisely $I = j^*(\tilde{I}) := \{a \in R : j(a) \in \tilde{I}\} = j^{-1}(\tilde{I}) \subseteq R$

preferred notation when working with $S^{-1}R$

In addition: $\tilde{I} = S^{-1}(j^*(\tilde{I}))$

Corollary 1: If $I = I_X$ for some $X \subseteq R$, then $S^{-1}(I)$ is the ideal of $S^{-1}R$ generated by $j(X)$.

Remark: Since $j : R \longrightarrow S^{-1}R$ is a ring homomorphism, we have:

$$\left\{\begin{array}{c}\text{Ideals in } R\\ \text{not intersecting } S\end{array}\right\} \underset{j^*}{\overset{j}{\rightleftarrows}} \left\{\begin{array}{c}\text{Ideals}\\ \text{of } S^{-1}R\end{array}\right\}$$

$$I \longmapsto S^{-1}I$$
$$R \supseteq j^*(\tilde{I}) \longleftarrow \tilde{I} \subset S^{-1}R$$

$j^*(S^{-1}I) \supseteq I$ because $a \in I \Rightarrow j(a) = \frac{a}{1} \in S^{-1}I \Rightarrow a \in j^*(S^{-1}I)$.

Q: Do we have $j^*(S^{-1}(I)) = I$ for any ideal $I \subseteq R$?

A: No! We'll see an example in §45.2.

Easy, if $I \cap S \neq \emptyset \Rightarrow S^{-1}I = S^{-1}R$ so $I \subseteq j^*(S^{-1}I) = j^*(S^{-1}R) = R$

Main result: We get a 1-to-1 correspondence if we take prime ideals

§45.2 More properties of ideals of $S^{-1}R$:

Proposition 3: Consider $I_1, I_2 \subseteq R$ ideals. Then:

(1) $S^{-1}(I_1 + I_2) = S^{-1}(I_1) + S^{-1}(I_2)$

(2) $S^{-1}(I_1 \cap I_2) = S^{-1}(I_1) \cap S^{-1}(I_2)$

(3) $S^{-1}(I_1 I_2) = S^{-1}(I_1)\, S^{-1}(I_2)$

Proof: (1) and (3) are obvious from Corollary 1 since $\begin{matrix} I_1 + I_2 \\ I_1 I_2 \end{matrix}$ is generated by $\begin{matrix} I_1 \cup I_2 \\ I_1 * I_2 \end{matrix}$

To prove (2), we check the double-inclusion.

($\subseteq$) $I_1 \cap I_2 \subseteq I_j$ for $j=1,2$ $\Rightarrow$ $S^{-1}(I_1 \cap I_2) \subseteq S^{-1}(I_j)$ for $j=1,2$

Thus $S^{-1}(I_1 \cap I_2) \subseteq S^{-1}(I_1) \cap S^{-1}(I_2)$

($\supseteq$) Pick $x \in S^{-1}(I_1) \cap S^{-1}(I_2) \Rightarrow x = \frac{a_1}{s_1} = \frac{a_2}{s_2}$ $\quad a_1 \in I_1,\ a_2 \in I_2,\ s_1, s_2 \in S$

Then: $\exists\, t \in S$ such that $t(a_1 s_2 - a_2 s_1) = 0 \Rightarrow \underset{I_1}{\overset{\cap}{a_1 t s_2}} = \underset{I_2}{\overset{\cap}{a_2 t s_1}} \in I_1 \cap I_2$

since $x = \frac{a_1 t s_2}{s_1 t s_2}$, we get $x \in S^{-1}(I_1 \cap I_2)$ $\square$

Proposition 4: Consider $I \subseteq R$ ideal & $S^{-1}(I) \subseteq S^{-1}R$ ideal. Then:

$$S^{-1}(I) = S^{-1}R \iff I \cap S \neq \emptyset$$

Proof: ($\Leftarrow$) If $s \in I \cap S$, then $1 = \frac{1}{s}\frac{s}{1} \in S^{-1}(I)$. Hence $S^{-1}(I) = S^{-1}R$.

($\Rightarrow$) Assume $S^{-1}(I) = S^{-1}R \Rightarrow \frac{1}{1} \in S^{-1}(I)$ ie $\exists\, a \in I\ s \in S$ with $\frac{1}{1} = \frac{a}{s}$

Thus $\exists\, t \in S$ with $t(1 \cdot s - a \cdot 1) = t(s-a) = 0$ So $ts = at$

$s, t \in S \Rightarrow st \in S$ because S is multiplicatively closed.

$a \in I,\ t \in S \subseteq R \Rightarrow at \in I$

} Conclude: $ts = at \in I \cap S$.

Hence $I \cap S \neq \emptyset$ $\square$

Proposition 5: For any ideal $I \subseteq R$ we have:

$$j^*(S^{-1}(I)) = \{ r \in R : tr \in I \text{ for some } t \in S \}$$

Proof: This is almost by definition:

$r \in j^*(S^{-1}(I)) \underset{\text{definition of } j^*}{\iff} \frac{r}{1} = j(r) \in S^{-1}(I) \iff \exists\, a \in I,\ s \in S$ with $\frac{r}{1} = \frac{a}{s}$

$\iff \exists\, s, s' \in S$ $a \in I$ with $s'(r \cdot s - a \cdot 1) = 0$, ie $\underbrace{r ss'}_{\substack{\shortparallel \\ t}} = s'a \in I$ This shows ($\subseteq$)

For (⊇) $\frac{r}{1} = \frac{tr}{t} \in S^{-1}(I)$ so $r \in j^*(S^{-1}(I))$.

Theorem: There exists a 1-to-1 correspondence:

$$\left\{\begin{array}{c}\text{Prime ideals in } R\\ \text{not intersecting } S\end{array}\right\} \xleftrightarrow{\text{1-to-1}} \left\{\begin{array}{c}\text{Prime ideals}\\ \text{of } S^{-1}R\end{array}\right\}$$

$$\mathfrak{P} \longmapsto S^{-1}(\mathfrak{P})$$

$$R \supsetneq j^*(\tilde{\mathfrak{P}}) \longleftarrow\!\shortmid\ \tilde{\mathfrak{P}} \subsetneq S^{-1}R$$

Proof: We first need to check these 2 maps are well-defined.

- **Note:** $\mathfrak{P} \subsetneq R$ & $\mathfrak{P} \cap S = \emptyset \Rightarrow S^{-1}(\mathfrak{P}) \subsetneq S^{-1}R$ is an ideal
  $\tilde{\mathfrak{P}} \subsetneq S^{-1}R \Rightarrow j^*(\tilde{\mathfrak{P}}) \subsetneq R$ because $j(1) = 1 \notin \tilde{\mathfrak{P}}$.
- Since $j: R \longrightarrow S^{-1}R$ is a ring homomorphism, we know by Proposition 2(2) §4.2,

that " $\tilde{\mathfrak{P}} \subsetneq S^{-1}R$ prime ideal $\Rightarrow \mathfrak{P} = j^*(\tilde{\mathfrak{P}}) \subsetneq R$ is also prime".

Claim 1: $\mathfrak{P} \subsetneq R$ prime ideal with $\mathfrak{P} \cap S = \emptyset \Rightarrow S^{-1}(\mathfrak{P}) = \{\frac{p}{s} : p \in \mathfrak{P}, s \in S\} \subsetneq S^{-1}R$ is a prime ideal

Pf/ We show $\tilde{a}, \tilde{b} \in S^{-1}R$ with $\tilde{a}\tilde{b} \in S^{-1}(\mathfrak{P}) \Rightarrow \tilde{a} \in S^{-1}(\mathfrak{P})$ or $\tilde{b} \in S^{-1}(\mathfrak{P})$.

Let $\frac{a_1}{t_1}, \frac{a_2}{t_2} \in S^{-1}R$, so $a_1, a_2 \in R$ & $t_1, t_2 \in S$.

Assume $\frac{a_1 a_2}{t_1 t_2} \in S^{-1}(\mathfrak{P})$ ie $\exists\, p \in \mathfrak{P}$ & $s \in S$ with $\frac{a_1 a_2}{t_1 t_2} = \frac{p}{s}$

Pick $t' \in S$ with $t'(s a_1 a_2 - p t_1 t_2) = 0$, ie $\underbrace{(t's)}_{\in S} a_1 a_2 = \underbrace{p t_1 t_2 t'}_{\in \mathfrak{P}} \in \mathfrak{P}$

Then $(t's)(a_1 a_2) \in \underset{\text{prime}}{\mathfrak{P}}$ & $t's \notin \mathfrak{P}$ $\Rightarrow a_1 a_2 \in \mathfrak{P}$
($t's \in S$, $S \cap \mathfrak{P} = \emptyset$)

Thus $a_1 \in \mathfrak{P}$ or $a_2 \in \mathfrak{P}$. Hence $\frac{a_1}{t_1} \in S^{-1}\mathfrak{P}$ or $\frac{a_2}{t_2} \in S^{-1}\mathfrak{P}$, as we wanted.

- Next, we check that the maps are inverse of each other.

We know from Proposition 2 that $S^{-1}(j^*(\tilde{I})) = \tilde{I}$ for every ideal $\tilde{I}$ of $S^{-1}R$.

Claim 2: $\mathfrak{P} \subsetneq R$ prime with $\mathfrak{P} \cap S = \emptyset \Rightarrow j^*(S^{-1}(\mathfrak{P})) = \mathfrak{P}$.

Pf/ We prove the double-inclusion:

($\subseteq$) By Proposition 5 §45.2, we have $r \in j^*(S^{-1}(\mathfrak{P})) \iff \exists\, t \in S$ with $tr \in \mathfrak{P}$.
But $S \cap \mathfrak{P} = \emptyset$, so $tr \in \mathfrak{P}$ with $t \in S \implies r \in \mathfrak{P}$.
(In fact: $\forall t \in S$: $tr \in \mathfrak{P} \iff r \in \mathfrak{P}$ because $t \notin \mathfrak{P}$ & $\mathfrak{P}$ is prime)

($\supseteq$) $p \in \mathfrak{P} \implies \frac{p}{1} \in S^{-1}(\mathfrak{P})$ so $\mathfrak{P} \subseteq j^*(S^{-1}(\mathfrak{P}))$ (This is true for any ideal I of R). $\square$

• We have similar results for arbitrary ideals I of R, but some important differences need to be stressed.

$$\begin{tikzcd} j: R \arrow[r] & S^{-1}R \\ j^*(\tilde{I}) \arrow[u, phantom, "\cup"] \arrow[r, dashed] & \tilde{I} \arrow[u, phantom, "\cup \text{ ideal}"] \end{tikzcd} \qquad \begin{tikzcd} j: R \arrow[r] & S^{-1}R \\ I \arrow[u, phantom, "\cup"] \arrow[r, mapsto] & S^{-1}(I) \arrow[u, phantom, "\cup \text{ ideal}"] \end{tikzcd}$$

$j^*(\tilde{I}) = \{r \in R : j(r) \in \tilde{I}\}$; I ideal

$$j^*(S^{-1}(I)) = \{r \in R : tr \in I \text{ for some } t \in S\} \supseteq I$$
$\hookrightarrow t = 1 \in S$

⚠ We may have $j^*(S^{-1}(I)) \supsetneq I$ even if $I \cap S = \emptyset$.
We know $S^{-1}(j^*(S^{-1}(I))) = S^{-1}(I)$ for every ideal $I \subseteq R$, so we will have two distinct ideals I & $j^*(S^{-1}(I))$ in R, not intersecting S which generate the same ideal of $S^{-1}R$. (This will forces I not to be prime)

<u>Example</u>: Take $R = K[x,y]$ with K a field.
$S = R \setminus (x) = \{f(x,y) \mid f \text{ is } \underline{\text{not}} \text{ divisible by } x\}$ is multiplicatively closed (because (x) is a prime ideal)
Take $I = (xy) \implies S^{-1}(I) = S^{-1}(xy) = S^{-1}(x)$
$S^{-1}R = \left\{ \frac{f(x,y)}{g(x,y)} \;\; g \text{ is not divisible by } x \right\}$
As $y \in S$, $y \cdot x \in I \implies x \in j^*(S^{-1}(I))$ but $x \notin I$.
<u>Exercise</u>: $j^*(S^{-1}(I)) = (x) \supsetneq (xy) = I$.