Lecture LII: Hilbert Basis Theorem

§52.1 Noetherian rings:

Recall: we defined commutative Noetherian rings as commutative rings satisfying either one of the following three properties:

(1) (Ascending Chain Condition) Given any ascending chain of ideals $I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$ we can find $n \geq 1$ so that $I_n = I_{n+1} = I_{n+2} = \cdots$ (ie, the chain stabilizes)

(2) Every non-empty set of ideals has at least one maximal element (with respect to inclusion)

(3) Every ideal is finitely generated.

Lemma: If $I = I_X$ (ie X is a generating set) and I is finitely generated, then $\exists\, Y \subseteq X$ finite such that $I = I_Y$. That is, the finite list of generators can be chosen among the input generating set.

Proof: Exercise.

§52.2 Being Noetherian vs on basic operations with rings:

Proposition Let R be a commutative ring. Assume that R is Noetherian.

(1) For any proper ideal $I \subsetneq R$, the quotient ring R/I is Noetherian

(2) For any multiplicatively closed set S of R, the ring of fractions $S^{-1}R$ is Noetherian.

(3) [easiest*] R_1, R_2 Noetherian $\Rightarrow$ $R_1 \times R_2$ is Noetherian.

Proof: (1) We use Proposition §51.2 & show every ideal of R/I is finitely generated.

Consider the natural projection $\pi : R \longrightarrow R/I$. By the Second Isomorphism Theorem §39.1, any ideal $\tilde{J} \subseteq R/I$ corresponds to the ideal $J = \pi^{-1}(\tilde{J}) \subseteq R$ which contains I.

Since R is Noetherian, we have $J = (a_1, \ldots, a_N)$ for some elements $a_1, \ldots, a_N$.

Then, $\tilde{J} \underset{\pi \text{ surjective}}{=} \pi(J) = (\pi(a_1), \ldots, \pi(a_N))$ so $\tilde{J}$ is finitely generated.

(2) As with (1) we show that every ideal of $S^{-1}R$ is finitely generated. We use the fact that we know what ideals of $S^{-1}R$ look like (Lemma 1 §44.2). Indeed, any ideal of $S^{-1}R$

has the form $S^{-1}I = \{ \frac{a}{s} : a \in I, s \in S \}$ for some ideal I of R.

Since I is finitely generated, say $I = (b_1, \dots, b_\ell)$, then we get

$S^{-1}I = \left(\frac{b_1}{1}, \dots, \frac{b_\ell}{1} \right)$ (generators $\{ j(b_1), \dots, j(b_\ell) \}$ with $j : R \longrightarrow S^{-1}R$)

$r \longmapsto \frac{r}{1}$

We conclude that $S^{-1}I$ is finitely generated.

(3) By the Lemma below, any ideal I of $R_1 \times R_2$ is of the form $I_1 \times I_2$ where $I_1 \subseteq R_1$ and $I_2 \subseteq R_2$ are ideals. Since I_1, I_2 are finitely generated, then so is I.

<u>Lemma</u>: Given R_1, R_2 rings, any ideal of $R_1 \times R_2$ is the cartesian product of ideals of R_1 & R_2.

<u>Proof</u>: We consider the natural projections

$$\begin{array}{ccc} R_1 \times R_2 & \xrightarrow{\pi_2} & R_2 \\ {\scriptstyle \pi_1} \downarrow & & \\ R_1 & & \end{array}$$

Both are surjections so given an ideal $I \subseteq R_1 \times R_2$, we obtain two ideals:

(1) $I_1 := \pi_1(I) \subseteq R_1$ & (2) $I_2 := \pi_2(I) \subseteq R_2$

<u>Claim</u>: $I = I_1 \times I_2$

Pf/ We prove the double inclusion.

($\subseteq$) We know $I \subseteq I_1 \times R_2 = \pi_1^{-1}(I_1)$ & $I \subseteq R_1 \times I_2 = \pi_2^{-1}(I_2)$. Thus, we get $I \subseteq I_1 \times I_2 = (I_1 \times R_2) \cap (R_1 \times I_2)$

($\supseteq$) Let $a_1 \in I_1$ & $a_2 \in I_2$. Then, $\exists\, b_1 \in R_1$ and $b_2 \in R_2$ such that

$a_1 = \pi_1((a_1, b_2))$ with $(a_1, b_2) \in I$,

$a_2 = \pi_2((b_1, a_2))$ with $(b_1, a_2) \in I$.

Then, $(a_1, a_2) = (1, 0) * (a_1, b_2) + (0, 1) * (b_1, a_2) \in I$ where $*$: mult in $R_1 \times R_2$.

Since a_1, a_2 are arbitrary, we conclude $I_1 \times I_2 \subseteq I$. $\square$

Subrings of Noetherian rings need not be Noetherian.

<u>Example</u>: Consider $R = K[x_1, x_2, x_3, \dots]$ where K is a field

We know R is not Noetherian (Example 2 §51.2)

Since R is an integral domain (because $K[x_1, \dots, x_N]$ is for all N), we can consider the multiplicatively closed set $S = R \setminus \{0\}$ and the ring of fractions $F(R) = S^{-1}R$

(it's called the field of fractions).

Then $j: R \hookrightarrow F(R)$ is an injective ring homomorphism. Thus, we get

- R is a subring of $F(R)$
- R is NOT Noetherian
- $F(R)$ is Noetherian because it is a field.

§52.3 Hilbert Basis Theorem:

Theorem: Assume R is commutative and Noetherian. Then, so is $R[x]$

In order the prove this result, we will need some notations and terms for polynomials $f \in R[x] \setminus \{0\}$:

$f(x) = a_0 + a_1 x + \cdots + a_{n-1} x^{n-1} + a_n x^n$ and assume n is the largest with $a_n \neq 0$.

- $n = \text{degree}(f)$ degree of $f(x)$
- $a_0 = f(0)$ constant term of $f(x)$
- $a_n = L(f)$ leading coefficient of $f(x)$.

Convention: if $f = 0$, then $\text{degree}(0) = -\infty$ & $L(0) = 0$

§52.4 Ideal of Leading Coefficients:

Let R be a commutative ring & let $\tilde{I} \subseteq R[x]$ be an ideal.

Consider the set $L(\tilde{I}) := \{a \in R : a = L(f) \text{ for some } f \in \tilde{I}\} \subset R$

Lemma: $L(\tilde{I}) \subset R$ is an ideal

Proof: (1) $0 = L(0)$ & $0 \in \tilde{I}$, so $0 \in L(\tilde{I})$

(2) If $a, b \in L(\tilde{I})$ $\exists$ $f, g \in \tilde{I}$ with $a = L(f)$ & $b \in L(g)$.

If $a+b = 0$, there is nothing to prove

If $a+b \neq 0$, assume $\deg(f) \leq \deg(g)$. Then: $a+b = L\big(\underbrace{f x^{\deg g - \deg f} + g}_{\in I}\big)$,

so $a+b \in L(\tilde{I})$.

(3) If $a \in L(\tilde{I})$ $a = L(f)$ fo $f \in \tilde{I}$. Then, $-a = -L(f) = L(-f)$ & $-f \in \tilde{I}$

so $-a \in L(\tilde{I})$.

(4) Let $a \in L(\tilde{I})$ with $a = L(f)$ $f \in \tilde{I}$ & $r \in R$. Then, assuming $ra \neq 0$ we have

$ra = L(rf)$ As $r \in R \subseteq R[x]$, $f \in \tilde{I}$, then $rf \in \tilde{I}$, so $ra \in L(\tilde{I})$.

If $ra = 0$, then $ra \in L(\tilde{I})$ automatically. $\square$

§52.5 Baby steps: towards a proof of Hilbert Basis Theorem:

Throughout, we assume R is a commutative ring.

Proposition: If R is a Noetherian ring and $D \in \mathbb{Z}_{\geq 1}$, then $R[x]/(x^D)$ is also Noetherian.

Remark: This statement in fact would follow from Hilbert Basis Theorem, but we are going to use it in our proof.

We will in fact prove a stronger result. Namely,

Lemma: Given an abelian subgroup $J \subseteq R[x]/(x^D)$ such that $R \cdot J \subseteq J$, we can find finitely many elements $f_1(x), \ldots, f_p(x) \in J$ such that

$$J = R \cdot f_1(x) + R \cdot f_2(x) + \cdots + R f_p(x)$$

Proof: For each $k \in \{0, \ldots, D-1\}$ we define

$$C_k(J) := \{0\} \cup \{a \in R \;:\; \exists f(x) \in J \text{ of the form } f(x) = a x^k + \sum_{j=k+1}^{D-1} a_j x^j\}$$

(ie collect lowest term of polynomials $f \in J$ with $x^k \mid f$.

Exercise: $C_k(J) \subset R$ is an ideal by our assumptions on J.

(use: $J \subset R[x]/(x^D)$ is a subgroup and $R \cdot J \subset J$)

As R is Noetherian, $C_k(J) = (\alpha_1^{(k)}, \ldots, \alpha_{m_k}^{(k)})$ for some finite number of elements $\alpha_1^{(k)}, \ldots, \alpha_{m_k}^{(k)} \in R$. These elements come from $f_1^{(k)}(x), \ldots, f_{m_k}^{(k)}(x) \in J$

That is
$$f_1^{(k)}(x) = \alpha_1^{(k)} x^k + \boxed{\begin{array}{l}\text{terms involving}\\ x^{k+1}, \ldots, x^{D-1}\end{array}} \in J$$
$$\vdots$$
$$f_{m_k}^{(k)}(x) = \alpha_{m_k}^{(k)} x^k + \boxed{\begin{array}{l}\text{terms involving}\\ x^{k+1}, \ldots, x^{D-1}\end{array}} \in J$$

Claim: $J = (R f_1^{(0)}(x) + \cdots + R f_{m_0}^{(0)}(x)) + (R f_1^{(1)}(x) + \cdots + R f_{m_1}^{(1)}(x)) +$
$\cdots + (R f_1^{(D-1)}(x) + \cdots + R f_{m_{D-1}}^{(D-1)}(x))$

Pf/ If $g(x) = \gamma x^\ell + \boxed{\begin{array}{l}\text{terms involving}\\ x^{\ell+1}, \ldots, x^{D-1}\end{array}} \in J$ then $\gamma \in C_\ell(J) = (\alpha_1^{(\ell)}, \ldots, \alpha_{m_\ell}^{(\ell)})$
$\quad\quad \cap$
$\quad\quad R$

This means $\gamma = r_1 \alpha_1^{(\ell)} + \dots + r_{m_\ell} \alpha_{m_\ell}^{(\ell)}$ for some $r_1, \dots, r_{m_\ell} \in R$

$\Rightarrow \bar{g}(x) = g(x) - \sum_{j=1}^{m_\ell} r_j f_j^{(\ell)} \in J$ and $\bar{g}(x) = \gamma' x^{\ell+1} + \boxed{\text{terms involving } x^{\ell+1}, \dots, x^{D-1}}$

So $\gamma' \in C_{\ell+1}(J)$.

Repeating this argument, after $D-\ell$ iterations of this arguments, we get the claim.

Remark: Any $J \subseteq R[x]/(x^D)$ ideal satisfies the conditions of the Lemma since $R \hookrightarrow R[x]/(x^D)$ is a subring. Thus, the Proposition follows directly from the Lemma.