ALGEBRA I (MATH 6111 AUTUMN 2020) - HOMEWORK 5

Problem 1. (Fun with commutators)

Let G be a group. For $a, b \in G$, define $[a, b] := aba^{-1}b^{-1}$. Recall that for any two subsets $A, B \subset G$, we defined (A, B) to be the subgroup generated by $\{[a, b] \colon a \in A, b \in B\}$.

(i) Verify the following identity, for all $a, x, y \in G$:

$$[a, xy] = [a, x][x, [a, y]][a, y].$$

(ii) Let A, B, C be three normal subgroups of G. Prove that $(A, (B, C))$ is generated by $\{[a, [b, c]] \colon a \in A, b \in B, c \in C\}$.

(iii) Recall that $C^1(G) = G$ and $C^{n+1}(G) := (G, C^n(G))$ defines the lower central series of G. Prove that for every $m, n \geq 1$ we have $(C^m(G), C^n(G)) \subseteq C^{m+n}(G)$.

Problem 2. Consider the following groups of matrices over $\mathbb{C}$.

$$B = \left\{ \begin{pmatrix} d_1 & x \\ 0 & d_2 \end{pmatrix} \text{ where } d_1, d_2 \neq 0 \text{ and } x \text{ is arbitrary} \right\}$$

$$N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \text{ where } x \text{ is arbitrary} \right\}$$

(i) Show that B is solvable.

(ii) Show that N and B/N are nilpotent, but B is not.

Problem 3. Let G be a group and let N_1, N_2 be two normal subgroups satisfying $(G, N_1) \subseteq N_2 \subseteq N_1$. Given any subgroup $H < G$, prove that $N_2 H \triangleleft N_1 H$.

Problem 4. Prove that the following three assertions about a finite group G are equivalent:

(i) G is nilpotent.

(ii) Every Sylow subgroup of G is normal.

(iii) G is a direct product of p-groups.

(*Hint:* Use the Lemma we saw in Lecture 14 and Problem 3 on Homework 3.)

Problem 5. Let G be a nilpotent group and let H be a proper subgroup of G. Prove that there exists a proper normal subgroup N of G, which contains H and such that G/N is abelian.
(*Hint:* See the proof of the Lemma of Lecture 14).

Problem 6. Let G be a nilpotent group and H be a subgroup. Prove that if $G = H.(G, G)$, then $H = G$. In other words, a subset X of G generates G if, and only if the image of X

under the natural surjection generates $G/(G,G)$.
(*Hint:* See the proof of the Lemma of Lecture 14).

Problem 7. Let $\varphi\colon G \to G'$ be a group homomorphism. Assume Σ' is a composition series of G':

$$\Sigma'\colon G' = G'_0 \triangleright G'_1 \triangleright G'_2 \triangleright \ldots \triangleright G'_n = \{e\}.$$

Let Σ be the sequence with terms $G_j = \varphi^{-1}(G'_j)$ for all $j = 0, \ldots, n$, and $G_{n+1} = \{e\}$.

(i) Prove that Σ is a composition series of G.

(ii) Prove that we have injective homomorphisms $\mathrm{gr}_i^{\Sigma}(G) \to \mathrm{gr}_i^{\Sigma'}(G')$ for each $0 \leq i \leq n-1$.

Problem 8. Let H be a group admitting a Jordan-Hoölder series. Let $\ell(H)$ be the number of terms in a Jordan-Hölder series of H. Show this number is well defined.

Problem 9. Let G be a group and N be a normal subgroup of G. Prove that G has a Jordan-Hölder series if, and only if both N and G/N do. In that case, prove that $\ell(G) = \ell(N) + \ell(G/N)$.

Problem 10. Compute the derived and lower central series of the symmetric groups S_2, S_3 and S_4.

Problem 11. Assume that G is a (non-trivial) nilpotent group. Prove that $\mathsf{Z}(G) \neq \{e\}$. Here, $\mathsf{Z}(G)$ is the center of G.
(*Recall:* G is nilpotent if and only if G admits a composition series $G = H_0 \triangleright \ldots \triangleright H_m = \{e\}$ such that $[G, H_\ell] \subset H_{\ell+1}$ for every ℓ.)

Problem 12. Fix a finite simple group S. For a finite group G, choose a Jordan-Hölder series $\Sigma : G = G_0 \triangleright G_1 \triangleright \ldots \triangleright G_n = \{e\}$. Let $\mathrm{Mult}(S;G)$ be defined as:

$$\mathrm{Mult}(S;G) := \#\{j : G_j/G_{j+1} \cong S\}$$

Prove that $\mathrm{Mult}(S;G)$ does not depend on the choice of the Jordan-Hölder series Σ.

Problem 12. Fix a finite simple group S. Let G be a finite group and $N \triangleleft G$ be a normal subgroup. Prove that $\mathrm{Mult}(S;G) = \mathrm{Mult}(S;N) + \mathrm{Mult}(S;G/N)$.