ALGEBRA I (MATH 6111 AUTUMN 2020) - HOMEWORK 6

Problem 1. Prove or disprove: the ring $R := \mathbb{Z}[x]/(x^2+1)$ is a principal ideal domain, that is R is a domain, and every ideal of R is generated by a single element.

Problem 2. Show that the ideal $\mathbb{Z}[x]$ generated by 2 and x is not principal.

Problem 3. (Radical of an ideal)
Fix a commutative ring A, and let $\mathfrak{a} \subset A$ be an ideal. Define

$$r(\mathfrak{a}) := \{a \in A : a^n \in \mathfrak{a} \text{ for some } n \in \mathbb{Z}_{\geq 1}\}.$$

Prove the following statements for two ideals $\mathfrak{a}, \mathfrak{b}$ of A:

(i) $r(\mathfrak{a})$ is an ideal of A.

(ii) $r(r(\mathfrak{a})) = r(\mathfrak{a})$.

(iii) $r(\mathfrak{a}\mathfrak{b}) = r(\mathfrak{a} \cap \mathfrak{b}) = r(\mathfrak{a}) \cap r(\mathfrak{b})$.

(iv) $r(\mathfrak{a} + \mathfrak{b}) = r(r(\mathfrak{a}) + r(\mathfrak{b}))$.

Problem 4. (Ideal quotients) Let $\mathfrak{a}$ and $\mathfrak{b}$ be two ideals of a commutative ring A. Set:

$$(\mathfrak{a} : \mathfrak{b}) := \{a \in A : a\mathfrak{b} \subseteq \mathfrak{a}\}.$$

(i) Prove that $(\mathfrak{a} : \mathfrak{b})$ is an ideal of A.

(ii) Compute the quotient ideal $((n) : (m))$ in $\mathbb{Z}$.

Problem 5. State and prove the three Isomorphism Theorems for modules over rings.

Problem 6. Let R be a ring, and M be a left R-module. Assume that we have $p \in \mathrm{End}_R(M) := \mathrm{Hom}_R(M, M)$ such that $p^2 = p$ (i.e. an *idempotent* map). Prove that $M \simeq M_1 \oplus M_2$, where M_1 is the kernel of p and M_2 is the image of p.

Problem 7. Give a counterexample to the assertion of Problem 6, if we do not impose the condition $p^2 = p$.

In the next two problems, we take $A = \mathbb{Z}$, $M = \mathbb{Z}/m\mathbb{Z}$ and $N = \mathbb{Z}/n\mathbb{Z}$.

Problem 8. Given $\alpha \in M$, define P_α to be the abelian group generated by two elements $\{e_1, e_2\}$, subject to the following relations:

$$m\, e_1 = 0 \qquad \text{and} \qquad n\, e_2 = \alpha\, e_1.$$

Verify that we have a natural short exact sequence:

$$\varepsilon_\alpha : \quad 0 \longrightarrow M \longrightarrow P_\alpha \longrightarrow N \longrightarrow 0 \ .$$

Problem 9. Determine the necessary and sufficient conditions for two short exact sequences ε_α and ε_β to be isomorphic (i.e. where we can build three vertical isomorphisms making the two squares commute).