ALGEBRA I (MATH 6111 AUTUMN 2020) - HOMEWORK 7

Problem 1. Let R be a commutative ring, $S \subseteq R$ a multiplicatively closed set. Show that the addition and multiplication operations on $S^{-1}R$ are well-defined and give $S^{-1}R$ a ring struture.

Problem 2. Consider $R = \mathbb{Z}/6\mathbb{Z}$, and let $S = \{1, 2, 4\}$. Compute $S^{-1}R$.

Problem 3. Let R be a commutative ring, $S \subseteq R$ a multiplicatively closed set and M an R-module. Consider the relation $\sim$ defined on $M \times S$:

$$(m, s) \sim (m', s') \quad \text{if, and only if,} \quad \exists\, t \in S \text{ with } t(s'\,m - s\,m') = 0 \in M.$$

(i) Show that $\sim$ defines an equivalence relation on $M \times S$.

(ii) Show that $S^{-1}M := (M \times S)/\sim$ is an $S^{-1}R$-module with operations:

$$(m, s) + (m', s') = (s'\,m + s\,m', ss') \quad \text{and} \quad (a, t)(m, s) = (a\,m, ts).$$

(iii) Prove that $S^{-1}M$ satisfies the universal property stated in Lecture 19.

Problem 4. (Nilradical of a commutative ring) Consider a commutative ring R and let $\mathcal{N} \subset R$ be the set of all nilpotent elements (see Lecture 18). Prove that

$$\mathcal{N} = \bigcap_{\substack{\mathfrak{p} \subset R \\ \mathfrak{p} \text{ prime ideal}}} \mathfrak{p}.$$

Problem 5. Fix a commutative ring R and let $\mathfrak{a} \subset R$ be the set of all zero-divisors of R. Is $\mathfrak{a}$ an ideal of R?

Problem 6. Let R be a commutative ring, $n \in R$ be a nilpotent element, and $u \in R$ a unit (that is, u has a multiplicative inverse). Prove that $u + n$ is again a unit.

Problem 7. Let R be a commutative ring, and let $S \subseteq R$ be a multiplicatively closed set. Let $\mathfrak{p}$ be an ideal which is maximal among the ideals in R not intersecting S. That is, maximal with respect to inclusion, from the following set:

$$I_S = \{\mathfrak{a} \subseteq R : \mathfrak{a} \text{ is an ideal of } R, \mathfrak{a} \cap S = \emptyset\}.$$

Prove that $\mathfrak{p}$ is prime.

Problem 8. Let A, B be commutative rings and let $f\colon A \to B$ be a ring homomorphism. Let $\mathfrak{p} \subset A$ be a prime ideal and define $\mathfrak{q} \subset B$ to be the ideal generated by $f(\mathfrak{p})$ in B. Prove or disprove: $\mathfrak{q}$ is a prime ideal.

Problem 9. (Jacobson radical)
Fix a commutative ring R and consider the following subset of R:

$$\mathfrak{J} := \{x \in R : 1 - xy \text{ is a unit for every } y \in R\}.$$

(i) Prove that $\mathfrak{J}$ is an ideal of R.

(ii) Prove that $\mathfrak{J} = \bigcap_{\substack{\mathfrak{m} \subset R \\ \mathfrak{m} \text{ maximal ideal}}} \mathfrak{m}$.

Problem 10. Consider the ring $R = \mathbb{C}[x]/(f)$, where $f \in \mathbb{C}[x]$ is a nonzero polynomial. Let us write $f(x) = \prod_{i=1}^{\ell} (x - a_i)^{n_i}$, where $a_1, \ldots, a_\ell \in \mathbb{C}$ and $n_1, \ldots, n_\ell \in \mathbb{Z}_{\geq 1}$. Prove that

$$R \xrightarrow{\sim} \prod_{i=1}^{\ell} \left(\mathbb{C}[x]/((x - a_i)^{n_i})\right) .$$

Problem 11. Let R be a commutative ring and fix a prime ideal $\mathfrak{p}$ of R. Prove that $r(\mathfrak{p}^n) = \mathfrak{p}$ for every $n \geq 1$. (Here, $r(\cdot)$ denotes the radical ideal, as defined in Problem 3 of Homework 6.)

Problem 12. Let R be a commutative ring. Fix $\mathfrak{m} \subset R$ a maximal ideal, and let $\mathfrak{p}$ be a prime ideal. Assume that there exists $n \geq 1$ such that $\mathfrak{m}^n \subseteq \mathfrak{p}$. Prove that $\mathfrak{m} = \mathfrak{p}$.

Problem 13. (Generalized prime avoidance)
Let R be a commutative ring and let $S \subset R$ be a set that is closed under multiplication and addition. Let $\mathfrak{p}_i \subset R$ for $i = 1, \ldots, n$, be a a finite list of ideals of R where at most two of them are not prime. Prove that if $S \subset \bigcup_{i=1}^{n} \mathfrak{p}_i$, then there exists $j = 1, \ldots, n$ with $S \subset \mathfrak{p}_i$. (*Hint:* Review the proof of the prime avoidance theorem.)

Problem 14. Let R be a commutative ring, $S \subset R$ a multiplicatively closed set and $\mathfrak{a} \subseteq R$ be an ideal. Prove that $S^{-1}\mathfrak{a}$ (the module of fractions of $\mathfrak{a}$ relative to S) is the ideal in $S^{-1}R$ generated by $j_S(\mathfrak{a})$, where $j_S \colon R \to S^{-1}R$ is the natural ring homomorphism.