# ALGEBRA I (MATH 6111 AUTUMN 2020) - HOMEWORK 8

In all problems below, we assume $R$ is a commutative ring.

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**Problem 1.** Consider the ring $R[x, y]$ and the multiplicatively closed set $S$ generated by $x$, that is $S = \{1, x, x^2, x^3, \ldots\}$. Prove the following isomorphisms of rings

$$(S^{-1})(R[x, y]/(xy)) \simeq R[x, x^{-1}].$$

**Problem 2.** Let $B$ be a commutative ring which contains $R$ as a subring. Assume that $B$ is finitely generated as a ring over $R$, that there exists finitely many elements $b_1, \ldots, b_\ell$ in $B$ such that every $b \in B$ can be written as a polynomial expression $\{b_1, \ldots, b_\ell\}$ with $R$ coefficients. Prove that, if $R$ is Noetherian, then so is $B$.

**Problem 3.** Assume $R[x]$ is Noetherian. Does it imply that $R$ is Noetherian?

**Problem 4.** Consider the following sequence of $R$-linear maps between $R$-modules:

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0.$$

(i) Prove that this sequence is exact if, and only if, for every maximal ideal $\mathfrak{m} \subset R$, the following sequence of $A_{\mathfrak{m}}$-modules is exact:

$$0 \longrightarrow M'_{\mathfrak{m}} \longrightarrow M_{\mathfrak{m}} \longrightarrow M''_{\mathfrak{m}} \longrightarrow 0.$$

(ii) Show that the same is true if we consider localizations at all prime ideals $\mathfrak{p}$ of $R$.

**Problem 5.** Assume that $R$ is not Noetherian. Let $\mathcal{S}$ be the set of all ideals of $R$ which are not finitely generated. Prove that this set has a maximal element and that any such maximal element is necessarily a prime ideal of $R$.

**Problem 6.** Assume that $R$ is Noetherian. Prove that so is $R[[x]]$ (i.e., power series with coefficients over $R$).

**Problem 7.** Assume that $R$ is Noetherian. Show that the nilradical $\mathcal{N}$ is a nilpotent ideal (i.e., there exists $k \in \mathbb{Z}_{\geq 1}$ with $\mathcal{N}^k = (0)$.)

**Problem 8.** Let $\mathfrak{p} \subsetneq R$ be a prime ideal of $R$. Show that the quotient $R_{\mathfrak{p}}/(\mathfrak{p}R_{\mathfrak{p}})$ is isomorphic to the quotient field $\mathrm{Quot}(A/\mathfrak{p})$.

**Problem 9.** Assume that $R_{\mathfrak{p}}$ is a Noetherian ring, for every prime ideal $\mathfrak{p}$ of $R$. Prove or disprove: $R$ is Noetherian.

**Problem 10.** Let $M$ be a Noetherian module over $R$. Consider the set $M[x]$, defined as:

$$M[x] = \{\sum_{i=0}^{N}(m_i x^i) : m_i \in M \text{ for all } i, N \geq 0\}.$$

(i) Show that $M[x]$ is an $R[x]$-module.

(ii) Show that $M[x]$ is a Noetherian module over $R[x]$.

(iii) What happens when we view $M[x]$ as an $R$-module?

**Problem 11.** Given an $R$-module $M$, we define the *support* of $M$ as:

$$\mathrm{Supp}(M) = \{\mathfrak{p} : \mathfrak{p} \text{ is a prime ideal of } R, M_{\mathfrak{p}} \neq (0)\}.$$

(i) If $N$ is a submodule of $M$, show that $\mathrm{Supp}(M) = \mathrm{Supp}(N) \cup \mathrm{Supp}(M/N)$.

(ii) Given an ideal $\mathfrak{a}$ of $R$, show that

$$\mathrm{Supp}(R/\mathfrak{a}) = \{\mathfrak{p} : \mathfrak{p} \text{ is a prime ideal of } R \text{ with } \mathfrak{p} \supseteq \mathfrak{a}\}.$$

**Problem 12.** Let $M$ be a Noetherian module over $R$. Let $f\colon M \to M$ be a surjective $R$-linear map. Prove that $f$ is an isomorphism. (*Hint:* consider the chain of submodules $\{\mathrm{Ker}(f^n)\}_{n\geq 0}$, where $f^0 = \mathrm{id}_M$.)