ALGEBRA I (MATH 6111 AUTUMN 2020) - HOMEWORK 9

In all problems below, we assume R is a commutative ring.

Problem 1. Let M be an Artinian R-module. That is, for every descending chain of submodules $M_0 \supseteq M_1 \supseteq M_2 \supseteq \ldots$, there exists $\ell \geq 0$ such that $M_\ell = M_{\ell+1} = \ldots$. Assume there is an injective R-linear map $f \colon M \to M$. Prove that f is an isomorphism.

Problem 2. Prove that a finite direct sum of Artinian rings is Artinian.

Problem 3. Show that an ideal $\mathfrak{q} \subset R$ is primary if, and only if, every zero divisor in $R/\mathfrak{q}$ is nilpotent.

Problem 4. Let $\mathfrak{m} \subset R$ be a maximal ideal. Prove that $\mathfrak{m}^n$ is primary for every $n \geq 1$.

Problem 5. Let $\mathfrak{q} \subset R$ be an ideal. If its radical ideal $r(\mathfrak{q})$ is maximal, then show that $\mathfrak{q}$ is primary.

Problem 6. Assume R is Noetherian and let $\mathfrak{p} \subset R$ be a prime ideal. Prove that $R_\mathfrak{p}$ is Artinian if, and only if $\mathfrak{p}$ is a minimal prime ideal of R.

Problem 7. Let $\mathfrak{a} \subset R$ be an ideal. Prove that

$$r(\mathfrak{a}) = \bigcap_{\mathfrak{p} \in \mathrm{Min}(\mathfrak{a})} \mathfrak{p}.$$

Problem 8. Assume that R is Noetherian. For any ideal I, prove that there exists $n \geq 1$ such that $r(I^n) \subset I$.

Problem 9. Let $I \subsetneq R$ a proper ideal, and $\mathfrak{p} \subsetneq R$ a prime ideal such that $I \subset \mathfrak{p}$. Prove that $r(I) \subset \mathfrak{p}$.

Problem 10. Prove the following equality is true in the ring $\mathbb{K}[x, y]$, where $\mathbb{K}$ is any field.

$$(x^2, y) \cap (x, y^2) = (x, y)^2.$$

Prove that $(x, y)^2 \subset \mathbb{K}[x, y]$ is a primary ideal. (*Hence, primary does not imply irreducible.*)

Problem 11. Let $R = \mathbb{K}[x, y]$, $I = (x^2, xy) \subset R$. Take $\mathfrak{p} = (x)$ and $\mathfrak{q}_n = (x^2, xy, y^n)$ for each $n \geq 2$. Prove that

(i) $\mathfrak{p}$ is a prime ideal. Each $\mathfrak{q}_n$ is primary and $r(\mathfrak{q}_n) = (x, y)$.
(ii) $\mathfrak{p} \cap \mathfrak{q}_n = I$.

Hence, we have infinitely many distinct primary decompositions. Notice that they all have the same set of primes $\{(x), (x, y)\}$.

Problem 12. Prove that $(4,t) \subset \mathbb{Z}[t]$ is a primary ideal. Verify that $r((4,t)) = (2,t)$ which is a maximal ideal in $\mathbb{Z}[t]$. Prove that $(2,t)^2 \subsetneq (4,t) \subsetneq (2,t)$. *Hence, a primary ideal need not be power of a prime.*

Problem 13. Consider a primary ideal $\mathfrak{q}$ in R with radical $r(\mathfrak{q}) = \mathfrak{p}$. Let $S \subsetneq R$ be a multiplicatively closed set.

(i) Prove that $S \cap \mathfrak{q} \neq \emptyset$ if, and only if, $S \cap \mathfrak{p} \neq \emptyset$.

(ii) Assume that $S \cap \mathfrak{p} = \emptyset$. Show that $S^{-1}\mathfrak{q}$ is a primary ideal in $S^{-1}R$ and $r(S^{-1}\mathfrak{q}) = S^{-1}\mathfrak{p}$ in $S^{-1}R$.

We say $\mathfrak{a} \subset \mathbb{K}[x_1, \ldots, x_n]$ is a **monomial ideal** if it is generated by monomials in the variables $x_1, \ldots, x_n$.

Problem 14. Characterize monomial ideals in $\mathbb{K}[x_1, \ldots, x_n]$ that are:

(i) prime;

(ii) irreducible;

(iii) radical (i.e., $r(\mathfrak{a}) = \mathfrak{a}$);

(iv) primary.

Bonus problem: Find algorithms for computing:

(i) the radical of a monomial ideal;

(ii) a primary decomposition of a monomial ideal.